Aczel's V and the notion of set

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Overview

Peter Aczel, "The type theoretic interpretation of constructive set theory",

- Logic Colloquium '77
- Brouwer centenary symposium, 1980

Interpretation of CZF in Martin-Löf's constructive type theory.

Why is it of interest to the philosopher?

- Sets are formed by a set-of operation.
- This yields a novel, pure iterative conception of sets.
- The relation between the notions of iterative set and type is clarified.
- Set equality is an equivalence relation on an underlying, more finely individuated domain.
- It gives constructive meaning to set theory.

Constructive set theory CZF

Similar to classical ZF, but with two main differences:

- Intuitionistic logic.
- Weaker forms of separation and power set axiom.

Nevertheless,

$$\mathrm{CZF} + \mathrm{LEM} = \mathrm{ZF}$$

Constructive Type Theory

- A formal system intended to serve as a foundation for constructive mathematics.
- Built up from first principles.
- Supplied with detailed meaning explanations, making it into a language in the proper sense (interpreted language).

Peter Aczel, "The type theoretic interpretation of constructive set theory":

> *I believe that the interpretation of constructive set theory in type theory can lay claim to give a good constructive meaning to the set theoretical notions.*

The interpretation of a theory T in constructive type theory provides a constructive justification of T.

The interpretation in rough outlines

- Constructive type theory is taken for granted.
  - In particular, we assume a range of types, including the natural numbers $\mathbf{N}$, to be available.
- Define a novel type $\mathbf{V}$ as the domain of the interpretation.
  - The type $\mathbf{V}$ will be inductively generated.
- Define a relation of extensional equality, $\doteq$, and a relation of membership, $\in$, between objects of type $\mathbf{V}$.
- Argue that the axioms of CZF are true in the structure $(\mathbf{V}, \doteq, \in)$.

Our interest is in the structure $(\mathbf{V}, \doteq, \in)$ and what it might teach us about the notion of set.

A simpler example: hereditarily finite sets

- Finite type theory is taken for granted.
- Define a novel type **HF** of the hereditarily finite sets.
  - The definition takes us out of finite type theory.
- Define a relation of extensional equality, $\doteq$, and a relation of membership, $\in$, between objects of type **HF**.

Finite type theory

Every type is finite, and for every n there is a type of cardinality n.

Formation rules:

$$\bot : \mathbf{Fin} \qquad \frac{A : \mathbf{Fin}}{S(A) : \mathbf{Fin}}$$

Introduction rules:

$$\bot \text{ is empty} \qquad *_A : S(A) \qquad \frac{a : A}{\sigma(a) : S(A)}$$

Define $\mathbf{0} = \bot$, $\mathbf{1} = S(\bot)$, $\mathbf{2} = S^2(\bot)$,..., $\mathbf{n} = S^n(\bot)$.

To define a function with domain $\mathbf{n}$ it suffices to specify its value for each of the n objects of $\mathbf{n}$.

For any type A, there is a function $R_A : \bot \to A$.

The set-of operation

- Given certain objects, application of the set-of operation produces the set of them.
- A precise account of the set-of operation requires a precise account of its operand.
- Naive set theory: the set-of operation applies to a propositional function P to form the set of objects a for which $P(a)$ is true.
- The "combinatorial" account: the set-of operation applies to a list of objects.
- Aczel: the set-of operation applies to a function f to form the set of objects enumerated by f.
  - This presupposes domains for the functions to be defined on. These are provided by type theory.

The hereditarily finite sets

The type **HF** of hereditarily finite sets is inductively generated by the following rule:

$$\frac{A : \mathbf{Fin} \qquad f : A \to \mathbf{HF}}{\mathbf{setof}(A, f) : \mathbf{HF}} \qquad (\mathbf{HF}\text{-intro})$$

Gloss: given a finite type A and a function $f : A \to \mathbf{HF}$, form the set of sets enumerated by f.

Alternative notation for $\mathbf{setof}(A, f)$:

$$\{f(a) \mid a : A\}$$

Applying the rule **HF**-intro

$$\frac{A : \mathbf{Fin} \qquad f : A \to \mathbf{HF}}{\mathbf{setof}(A, f) : \mathbf{HF}}$$

The empty set, $\emptyset$, is formed from $R_{\mathbf{HF}} : \bot \to \mathbf{HF}$.

Given sets $a_1, \ldots, a_n$, these can be enumerated by a function $f : \mathbf{n} \to \mathbf{HF}$. Then

$$\mathbf{setof}(\mathbf{n}, f) = \{a_1, \ldots, a_n\}$$

Inductive definition

In Kleene's terminology, (**HF**-intro) provides a *fundamental* inductive definition, by means of which a domain of objects is introduced.

Another example: the natural numbers.

$$0 : \mathbf{N} \qquad \frac{n : \mathbf{N}}{\mathbf{s}(n) : \mathbf{N}} \qquad (\mathbf{N}\text{-intro})$$

On any inductively defined domain, functions may be defined by induction.

In Kleene's terminology, such definitions are *non-fundamental*.

Strict identity

The criterion of identity for the type **HF** is the following:

$$\textbf{setof}(A, f) \equiv \textbf{setof}(B, g) \quad \text{iff} \quad A \equiv B \text{ and } f \equiv g$$

This definition accords with the usual pattern of defining *criterial identity* in constructive type theory.

But it is not a definition of extensional equality.

Let $f : \mathbf{1} \to \mathbf{HF}$ be constantly $\emptyset$.
Let $g : \mathbf{2} \to \mathbf{HF}$ be constantly $\emptyset$.

Then f and g enumerate the same sets (namely $\emptyset$), but $\textbf{setof}(\mathbf{1}, f)$ and $\textbf{setof}(\mathbf{2}, g)$ are not $\equiv$-identical.

Extensional equality

Extensional equality, $\doteq$, on **HF** is defined by induction:

$$\mathbf{setof}(A, f) \doteq \mathbf{setof}(B, g) \text{ iff } \begin{array}{l}(\forall x : A)(\exists y : B) f(x) \doteq g(y) \ \& \\ (\forall y : B)(\exists x : A) f(x) \doteq g(y)\end{array}$$

Gloss: provided $f(a) \doteq g(b)$ has been defined for arbitrary $a : A$ and $b : B$, define $\mathbf{setof}(A, f) \doteq \mathbf{setof}(B, g)$ as:

> *For every $f(a)$ there is a $g(b)$ such that $f(a) \doteq g(b)$, and for every $g(b)$ there is a $f(a)$ such that $f(a) \doteq g(b)$.*

This is a double induction on the build-up of objects of **HF**.

Bisimulation

Let $\prec$ be the immediate predecessor relation, $f(a) \prec \textbf{setof}(A, f)$.

$$u \doteq v \quad \text{iff} \quad \text{for each } x \prec u \text{ there is } y \prec v \text{ such that } x \doteq y, \text{ and}$$
$$\text{for each } y \prec v \text{ there is } x \prec u \text{ such that } x \doteq y.$$

Technical terminology: $\doteq$ is the largest bisimulation on $(\mathbf{HF}, \prec)$.

Historical note: Aczel would use bisimulation also in defining equality of non-well-founded sets.

Membership

Membership, $\in$, is explicitly defined:

$$u \in v \quad \text{iff} \quad \text{there is } x \prec v \text{ such that } u \doteq x$$

The following implications are then easily verified:

$$\begin{aligned} u \in v \ \&\ v \doteq w &\implies u \in w \\ u \in v \ \&\ u \doteq w &\implies w \in v \end{aligned}$$

This completes the definition of the structure $(\mathbf{HF}, \doteq, \in)$.

Aczel's V

Aczel's **V** is defined precisely as **HF** is defined with the sole difference that many more types are allowed to serve as domain of the enumerating functions.

$$\frac{A : \mathbf{U} \qquad f : A \to \mathbf{V}}{\mathbf{setof}(A, f) : \mathbf{V}} \qquad (\mathbf{V}\text{-intro})$$

U is a type of types:

$\bot$, $\mathbf{1}$

$\mathbf{N}$

$A + B$

$\Pi(A, C)$, $\Sigma(A, C)$

Dedekind infinity of **HF** and **V**

On both **HF** and **V** one can define a successor function,

$$\mathbf{suc}(x) \equiv \text{“} x \cup \{x\} \text{”}$$

Extensional

$$x \doteq y \implies \mathbf{suc}(x) \doteq \mathbf{suc}(y)$$

Injective

$$\mathbf{suc}(x) \doteq \mathbf{suc}(y) \implies x \doteq y$$

Not surjective

$$\neg(\mathbf{suc}(x) \doteq \emptyset)$$

The function **suc** thus witnesses that $(\mathbf{HF}, \doteq)$ and $(\mathbf{V}, \doteq)$ are Dedekind infinite.

Axiom of infinity

Define $N : \mathbf{N} \to \mathbf{V}$

$$\begin{aligned} N(0) &\equiv \emptyset \\ N(\mathbf{s}(n)) &\equiv \mathbf{suc}(N(n)) \\ &= N(n) \cup \{N(n)\} \end{aligned}$$

Let $\omega \equiv \mathbf{setof}(\mathbf{N}, N)$.

Then ω witnesses that the Axiom of Infinity is true in the structure $(\mathbf{V}, \doteq, \in)$,

$\emptyset \in \omega$

$n \in \omega \implies \mathbf{suc}(n) \in \omega$

Remarks on the interpretation

- The type **U** is a universe: an inductively defined type of (codes of) types.
- The types **HF** and **V** are examples of a so-called well-ordering type, $\mathbf{W}(A, C)$.
- Under the Curry–Howard correspondence, **U** is also a type of propositions. This is used in the validation of the Axiom of Restricted Separation.
- The validation of the so-called collection axioms uses Intensional Axiom of Choice, a theorem of type theory.
- If Extensional Axiom of Choice is added to type theory, all of ZFC is validated in $(\mathbf{V}, \doteq, \in)$.

Set versus type

The priority of type over sets:

1. The domains of the enumerating functions in ($\mathbf{V}$-intro) are types.
2. $\mathbf{V}$ itself is a type.

The set-theoretic universe, $\mathbf{V}$, is just one type among (infinitely) many: $\mathbf{N}$, $\perp$, $\mathbf{N} \to \mathbf{N}$, ...

This is not a reduction of sets to types. Rather we understand sets as the objects of a certain inductively defined type $\mathbf{V}$.

Sets as individuals

Simple type theory, as defined by Church, has just one type, ι, of individuals.

Constructive type theory is many-sorted: (infinitely) many types of individuals (= sorts).

V is one such type.

Sets are thus rendered as individuals.

Iterative set

On a standard account, an iterative set is a member of the cumulative hierarchy.

The cumulative hierarchy is defined in terms of the power set operation and the classical notion of ordinal number.

An object in $(\mathbf{V}, \doteq)$ is obtained solely by iterating the set-of operation, starting from the empty set: no need to appeal to a power set operation.

It thus provides a purer account of the notion of an iterative set.

Note: Replacement is validated in $(\mathbf{V}, \doteq, \in)$.

$\mathbf{V}$ versus $(\mathbf{V}, \doteq)$

The type $\mathbf{V}$ is equipped with strict (intensional) identity, $\equiv$.

As such it is not a type of sets (no axiom of extensionality).

Sets are objects of the structure $(\mathbf{V}, \doteq)$, where $\doteq$ plays the role of identity.

Similar contrast: **Cauchy** versus $(\mathbf{Cauchy}, \sim)$, where $\sim$ is the relation of co-convergence.

A structure of the form (A, R), where R is an equivalence relation on A, is often called a *setoid*.

$(\mathbf{V}, \doteq)$ as an extensional domain

The criterion of identity in $(\mathbf{V}, \doteq)$ is given by a propositional function on $\mathbf{V}$.

- A propositional function, being a function, needs a domain of definition.
- Such a domain must be associated with a criterion of identity.
- Hence not every criterion of identity can be given by a propositional function.

It is natural to call $\mathbf{V}$ a fundamental domain and $(\mathbf{V}, \doteq)$ a derived domain.

A derived domain is obtained by extensionalization of a fundamental domain, where identity is intensional.