Brouwer's Notion of Choice Sequence and Its Descendants

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Intensionality and Extensionality in Mathematics I

Mathematics deals with ideal objects: numbers, sets, functions, ...

The natural numbers are defined or constructed inductively, starting with 0 and iterating the successor operation $S(\cdot)$:

$$0, S(0), S(S(0)), S(S(S(0))), \ldots$$

or in the less cumbersome decimal notation: $0, 1, 2, 3, \ldots$

The terms 2 and $S(S(0))$ have the same intension, the number itself, which is also their common extension. Thus $2 \equiv S(S(0))$ (where $\equiv$ denotes intensional equality) and also $2 = S(S(0))$.

The integers and rational numbers can be “coded” constructively by defining one-to-one correspondences with the natural numbers.

Intensionality and Extensionality in Mathematics II

In 1994 Andrew Wiles proved *Fermat's Last Theorem*: The largest natural number n for which there exist positive natural numbers x, y, z such that $x^n + y^n = z^n$ is $n = 2$.

In this case the equality is (only) extensional, asserting that the number 2 is the extension of the descriptive phrase

" the largest natural number n for which there exist positive natural numbers x, y, z such that $x^n + y^n = z^n$ "

whose intension was clear for centuries before it was established that the marvelous theorem Pierre Fermat conjectured around 1630 was correct, or even that such a number n exists.

Potential vs. Completed Infinity in Mathematics I

The standard natural numbers 0, 1, 2, 3, . . . constitute the smallest collection $\mathbb{N}$ containing 0 and closed under the successor operation.

In classical mathematics, $\mathbb{N}$ is a completed infinite totality, which is assumed to satisfy the

Principle of (Complete) Mathematical Induction: If $A(n)$ is any property such that (i) $A(0)$ holds, and (ii) whenever $n \in \mathbb{N}$ and $A(n)$ holds then $A(S(n))$ holds, then $A(n)$ holds for every $n \in \mathbb{N}$.

In intuitionistic mathematics, $\mathbb{N}$ is a potentially infinite totality, and the Principle of Mathematical Induction needs no justification. It is intuitively clear from the construction of tne natural numbers.

Brouwer's early treatment of the continuum

In his 1907 dissertation Brouwer wrote that the rational numbers, together with those irrational numbers like $\sqrt{2}$ which are definable using rationals, form a potentially infinite, denumerable, dense "scale" of order type η which does not exhaust the continuum.

Brouwer wrote "Mathematics can deal with no other matter than that which it has itself constructed;" and "*all* or *every* ... tacitly involves the restriction: *insofar as belonging to a mathematical structure which is supposed to be constructed beforehand*."

In order to complete a scale of order type η to a "measurable continuum, ... a 'matrix of points to be thought of as a whole'," he had to rely on a primitive intuition of continuity or "fluidity."

The "Second Act of Intuitionism"

By 1918-19 Brouwer had developed a new view of the continuum. In "Historical background, principles and methods of intuitionism" (1952) he recalled that the "Second Act of Intuitionism" explicitly recognized "the possibility of generating new mathematical entities:

"firstly in the form of infinitely proceeding sequences $p_1, p_2, \ldots$ whose terms are chosen more or less freely from mathematical entities previously acquired . . . ;

"secondly in the form of mathematical species, i.e. properties supposable for mathematical entities previously acquired, and satisfying the condition that, if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be equal to it . . ."

Potential vs. Completed Infinity in Mathematics II

A function f from a set or collection X to a set Y is defined as

- a rule or correspondence assigning to each element x of its domain X exactly one element $f(x) = y$ of its codomain Y.

The range of a function f with domain X and codomain Y consists of those elements of Y which are assigned by f to elements of X.

In intuitionistic mathematics, a function f from X to Y is a rule or correspondence such that, if X is potentially infinite, then

- the rule f may be "lawlike" in the sense that as soon as $x \in X$ is produced, $f(x)$ is completely determined,
- or the assignment of $y \in Y$ to $x \in X$ may proceed more or less freely in parallel with the generation of elements of X, and possibly also with the generation of elements of Y.

Brouwer's Infinitely Proceeding Sequences

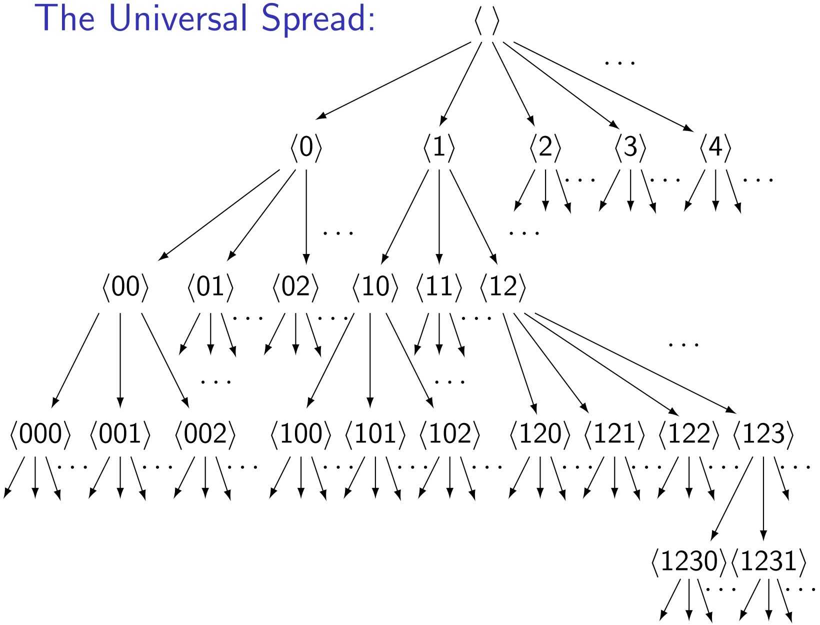
A choice sequence or infinitely proceeding sequence of natural numbers is a function with domain and codomain $\mathbb{N}$. Lower case Greek letters denote choice sequences $\alpha : \mathbb{N} \to \mathbb{N}$.

The universal spread is the potentially infinite collection of *all these infinitely proceeding sequences, in process of generation*. It can be visualized as a rooted tree, with nodes labeled by finite sequences of numbers representing initial segments of choice sequences:

$$\overline{\alpha}(0) = \langle \, \rangle, \qquad \overline{\alpha}(S(n)) = \langle \alpha(0), \alpha(1), \ldots, \alpha(n) \rangle$$

Brouwer considered general or arbitrary choice sequences, which may allow complete or partial freedom in the assignment of natural numbers $\alpha(n)$ to natural numbers n, and lawlike sequences (choice sequences α for which every choice $\alpha(n)$ is predetermined).

The Universal Spread:

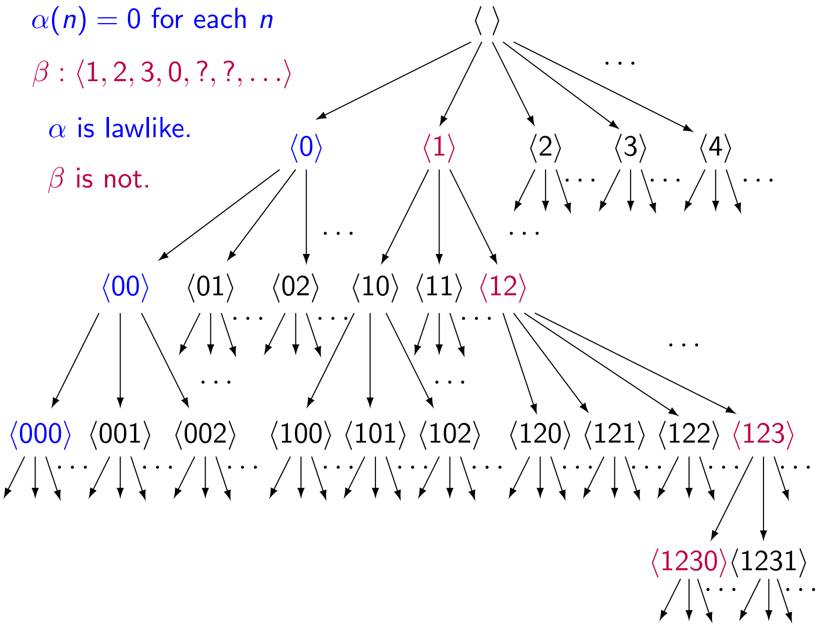


$\alpha(n) = 0$ for each n

$\beta : \langle 1, 2, 3, 0, ?, ?, \ldots \rangle$

α is lawlike.

β is not.



Brouwer's "Bar Theorem" is a Bar Induction Axiom

Intuitively, a "bar" on the universal spread is a set (or "species") X of nodes containing at least one node from every choice sequence: $\forall\alpha\exists x(\overline{\alpha}(x) \in X)$. Brouwer thought he had proved that if X is a bar on the universal spread and A is a property of nodes such that

1. A holds on every node ξ in X, and
2. for every node ζ of the universal spread: if A holds at every immediate successor of ζ, then A holds also at ζ,

then A holds at the root $\langle\,\rangle$.

Kleene gave a counterexample to show that this version conflicts with Brouwer's own continuity principle, and corrected it by requiring X to contain *just one* node from each choice sequence. The conflict also disappears when X is a *monotone* bar satisfying $\forall\alpha(\overline{\alpha}(x) \in X \rightarrow \overline{\alpha}(S(x)) \in X)$. But the "theorem" is an axiom.

Brouwer's Lawlike Sequences: What Are They?

Lawlike sequences of natural numbers (like the sequence of digits in the decimal expansion of $\sqrt{2}$) are definable, so there are only *denumerably* many; but there is no lawlike enumeration of them all.

Are they computable? Brouwer did not affirm Church's Thesis, but every sequence given by a primitive recursive definition, or by a Gödel number e with a proof that $\{e\}$ is total, should be lawlike.

As early as 1945, Kleene saw a parallel between Brouwer's lawlike sequences and (classical) general recursive functions. He read Brouwer carefully, and showed that the (corrected) "Bar Theorem" did *not* hold if *all* choice sequences were assumed to be recursive.

With this in mind, he began to axiomatize Brouwer.

Intensionality and Extensionality in Mathematics III

In mathematics, every set X comes with a binary relation of equality between its elements, typically written "$x = y$."
In classical mathematics, equality is always decidable: for every set X and for all $x, y \in X$, either $x = y$ or $x \neq y$ but not both.

In intuitionistic mathematics,

- Intensional equality $\equiv$ is always decidable.
- Extensional equality of natural numbers is decidable:
$$\forall n, m \in \mathbb{N} \, [n = m \vee n \neq m].$$
- Extensional equality of choice sequences is not decidable:
$$\neg \forall \alpha, \beta \in \mathbb{N}^{\mathbb{N}} \, [(\alpha = \beta) \vee \neg(\alpha = \beta)],$$
where $\mathbb{N}^{\mathbb{N}}$ is the spread of all infinitely proceeding sequences with extensional equality $(\alpha = \beta)$ defined by $\forall n(\alpha(n) = \beta(n))$.

Kleene's and Vesley's Approach to Formalizing Brouwer

In *The Foundations of Intuitionistic Mathematics, Especially in Relation to Recursive Functions* (1965) Kleene and Vesley developed an intuitionistic formal system **I** with number variables $x, y, \ldots$; choice sequence variables $\alpha, \beta, \ldots$; constants $0, S, \ldots, \mathrm{f_k}$ for primitive recursive functions; $=$ for numbers; Church's λ; and the extensional equality axiom: $(x = y \rightarrow \alpha(x) = \alpha(y))$.

The neutral subsystem **B** of **I** is a two-sorted extension of intuitionistic arithmetic, with axiom schemas for countable choice and Brouwer's "Bar Theorem". The full intuitionistic system **I** adds to **B** a classically false schema of continuous choice. Kleene provided a consistency proof for **I** using recursive realizability.

He conjectured, and in (1969) proved, that if $\exists\alpha A(\alpha)$ is closed and $\mathbf{I} \vdash \exists\alpha A(\alpha)$ then for some $\mathbf{e}$: $\mathbf{I} \vdash \exists\alpha[A(\alpha)\ \&\ \forall x(\alpha(x) = \{\mathbf{e}\}(x))]$.

Kreisel's and Troelstra's Alternative Approach I

In "Formal systems for some branches of intuitionistic analysis" (*Annals of Mathematical Logic*, 1970) Kreisel and Troelstra presented an alternative to Kleene's and Vesley's **I**.

EL ("elementary analysis") is a two-sorted "lawlike" extension of intuitionistic arithmetic, with number variables $x, y, z, \ldots$, and "variables for (constructive) functions (denoted by a, b, c, d)"; constants for (all) primitive recursive functions and λ-abstraction; and countable choice for quantifier-free relations – admitting that all general recursive functions are lawlike!

Note: Finite sequences of natural numbers are coded primitive recursively. In **EL** every x codes a sequence whose length $\leq x$ is recoverable from x. $\langle\,\rangle = 0$ codes the empty sequence; $\langle x \rangle$ codes the sequence consisting of just x; and $*$ denotes concatenation.

Kreisel's and Troelstra's Alternative Approach II

Kleene expressed Brouwer's "Bar Theorem" by an axiom schema. Instead, $\mathbf{IDB}_1$ adds to the language of **EL** a constant K for the inductively generated class of lawlike neighborhood functions of continuous functionals of type 1, and adds to the axioms of **EL**

K1. $K(\lambda n.(x+1))$.

K2. $a(\langle\,\rangle) = 0 \;\&\; \forall x\, K(\lambda n.a(\langle x\rangle * n)) \rightarrow K(a)$.

K3. $\forall a\, (A(Q,a) \rightarrow Q(a)) \rightarrow \forall a\, (K(a) \rightarrow Q(a))$
for all formulas Q of the language, where
$A(Q,a) \equiv \exists x(a = \lambda n.x+1) \vee (a(0) = 0 \;\&\; \forall x Q(\lambda n.a(\langle x\rangle * n)))$.

$\mathbf{IDB}_1$ also has a stronger axiom schema of countable choice:

AC_{01}. $\forall x \exists a\, A(x,a) \rightarrow \exists b \forall x\, A(x, \lambda y.b(j(x,y)))$.

The constant j represents a primitive recursive pairing function.

Kreisel's Lawless Sequences and Troelstra's Modification

$\mathbf{IDB}_1 = \mathbf{EL} + \text{K1-3} + \text{AC}_{01}$ only talks about lawlike sequences, while Kleene's **B** and **I** talk about arbitrary choice sequences.

Kreisel defined a lawless sequence of natural numbers to be a choice sequence admitting *no* restrictions; at each stage of its generation, only a finite initial segment has been determined, and every natural number is eligible to be chosen next.

Kreisel's and Troelstra's formal systems **LS** and **CS** are three-sorted intuitionistic extensions of $\mathbf{IDB}_1$, with variables x,y,z,... over numbers, a,b,c,... over lawlike sequences, and $\alpha, \beta, \gamma, \ldots$ over lawless sequences (for **LS**) or choice sequences (for **CS**).

$\mathbf{LS} = \mathbf{IDB}_1 + \text{L1-4}$ and

$\mathbf{CS} = \mathbf{IDB}_1 + \text{GC1-4}$, where

Axioms for Lawless Sequences as Improved by Troelstra

L1. $\forall n \exists \alpha (\alpha \in n)$ is the *density* axiom.

The next axiom says that (not only intensional, but also) extensional equality of lawless sequences is decidable.

L2. $\forall \alpha \forall \beta (\alpha \neq \beta \vee \alpha = \beta)$.

To express *relative independence* of lawless variables Troelstra defined quantifiers $\dot{\forall}\alpha, \dot{\exists}\alpha$ so that e.g. $\dot{\forall}\alpha A(\alpha, \beta, \gamma)$ is equivalent to $\forall \alpha (\alpha \neq \beta \,\&\, \alpha \neq \gamma \rightarrow A(\alpha, \beta, \gamma))$, and if $\vec{\alpha} = \alpha_0, \ldots, \alpha_k$ then $\dot{\forall}\vec{\alpha} A(\vec{\alpha})$ expresses $\forall \alpha_0 \ldots \forall \alpha_k (\forall i < j \leq k (\alpha_i \neq \alpha_j) \rightarrow A(\vec{\alpha}))$.

With all lawless parameters shown, the schema of open data is

L3. $\dot{\forall}\alpha (A(\alpha, \vec{\beta}) \rightarrow \exists n (\alpha \in n \,\&\, \dot{\forall}\gamma \in n \, A(\gamma, \vec{\beta})))$

and the bar continuity schema (with lawlike e and b) is

L4. $\dot{\forall}\vec{\alpha} \exists b A(\vec{\alpha}, b) \rightarrow \exists e (K(e) \,\&\, \forall n (e(n) \neq 0 \rightarrow \exists b \dot{\forall}\vec{\alpha} \in n A(\vec{\alpha}, b)))$

Kreisel and Troelstra's "elimination of lawless sequences"

In (1968) Kreisel proved the *"first elimination theorem"*: Every formula with no free lawless sequence variables is equivalent in **LS** to one with no lawless sequence variables. In (1969) Troelstra improved Kreisel's result to the *"second elimination theorem"*:

The elimination of lawless sequences holds for **LS** in $\mathbf{IDB}_1$:
There is a syntactic translation τ mapping each formula E of the language of **LS** without free lawless sequence variables to a formula $\tau(E)$ without any lawless sequence variables such that

(i) $\vdash_{\mathbf{LS}} (E \leftrightarrow \tau(E))$.

(ii) $\vdash_{\mathbf{LS}} E \Leftrightarrow \vdash_{\mathbf{IDB_1}} \tau(E)$.

(iii) $\tau(E) \equiv E$ if E has no lawless sequence variables.

Corollary: **LS** is a conservative extension of $\mathbf{IDB}_!$.

The translation τ which gradually eliminates quantifiers over lawless sequences from formulas in **LS** is complex. It involves e.g.

- using LS2 to replace $\forall\alpha A(\alpha,\beta)$ by $(A(\beta,\beta)\ \&\ \dot\forall\alpha A(\alpha,\beta))$,
- using LS2 to replace $\exists\alpha A(\alpha,\beta)$ by $(A(\beta,\beta) \vee \dot\exists\alpha A(\alpha,\beta))$,
- replacing $A \vee B$ by $\exists n((n = 0 \rightarrow A)\ \&\ (n \neq 0 \rightarrow B))$,
- replacing $\dot\exists\alpha A(\alpha,\beta)$ by $\exists n \dot\forall\alpha \in n\, A(\alpha,\beta)$ using LS3,
- moving $\dot\forall\alpha \in n$ to the inside using LS4, which introduces new number and lawlike sequence quantifiers only, and
- replacing $\dot\forall\alpha \in n(s(\alpha) = t(\alpha))$ by $\forall a \in n(s(a) = t(a))$.

Lawless Sequences and Troelstra's Extension Principle

In $\mathbf{IDB_1}$ a class K of monotone neighborhood functions is defined *inductively* by axioms K1-3. Is there an *explicit* definition? If

$K_0(e) \equiv \forall a \exists n\, e(\overline{a}(n)) \neq 0 \;\&\; \forall m \forall n (e(n) > 0 \rightarrow e(n) = e(n * m))$

then $\vdash_{\mathbf{IDB_1}} \forall e(K(e) \rightarrow K_0(e))$ but $\nvdash_{\mathbf{IDB_1}} \forall e(K_0(e) \rightarrow K(e))$, and

Brouwer's Bar Theorem fails for the lawlike sequences.

But $\vdash_{\mathbf{LS}} \forall e(K(e) \rightarrow \forall \alpha \exists x\, e(\overline{\alpha}(x)) \neq 0)$ so if $K_0^*(e)$ is like $K_0(e)$ but with α in place of a then $\vdash_{\mathbf{LS}} \forall e(K_0^*(e) \rightarrow K(e))$ using LS4.

Troelstra argued that the initial segments of *any* choice sequence can be viewed as initial segments of a lawless sequence in process of generation. In Brouwer's terminology, Troelstra's Extension Principle states that every bar on the lawless sequences bars *all* sequences of natural numbers, so the Bar Theorem holds for **LS**.

General Choice Sequences: the Principle of Analytic Data

CS $= \mathbf{IDB}_1 +$ GC 1-4 has the same language as **LS**. Adapting Kleene, $e|\alpha = \beta$ abbreviates $\forall y(\lambda n.e(\langle y \rangle * n))(\alpha) = \beta(y))$ where $e(\alpha) = t$ abbreviates $\exists x\, e(\overline{\alpha}(x)) = t + 1$. The new axioms are

Closure under lawlike continuous functions (which *fails* for **LS**):
GC1. $\forall e(K(e) \to \forall\alpha\exists\beta(e|\alpha = \beta))$ and $\forall\alpha\forall\beta\exists\gamma(j(\alpha, \beta) = \gamma)$,

Troelstra's Principle of Analytic Data:
GC2. $A(\alpha) \to \exists e(K(e)\ \&\ \exists\beta(e|\beta = \alpha)\ \&\ \forall\beta A(e|\beta))$,

and Kleene's continuous choice axioms with a lawlike modulus:
GC3. $\forall\alpha\exists b A(\alpha, b) \to \exists e\forall n(e(n) \neq 0 \to \exists b\forall\alpha \in n A(\alpha, b))$,
GC4. $\forall\alpha\exists\beta\, A(\alpha, \beta) \to \exists e\forall\alpha\, A(\alpha, e|\alpha)$.

- The elimination of choice sequences holds for **CS** in $\mathbf{IDB}_1$,
- $\vdash_{\mathbf{CS}} \forall e(K_0^*(e) \leftrightarrow K(e))$ (the monotone bar theorem), and
- $\vdash_{\mathbf{CS}} \forall\alpha\neg\neg\exists b(\alpha = b)$. (In contrast, $\vdash_{\mathbf{LS}} \forall\alpha\neg\exists b(\alpha = b)$.)

Other Work on Choice Sequences

In e.g. "A classical view of the intuitionistic continuum" (Ann. Math. Logic, 1996), I defined and studied a notion of "relatively lawless sequence." A *predictor* is a function from finite sequences of natural numbers to finite sequences of natural numbers. A choice sequence α is lawless relative to a given notion of lawlike sequence if every lawlike predictor correctly predicts some segment of values of α on the basis of the values already chosen. The relatively lawless sequences satisfy versions of the axioms of **LS** but *not* the Bar Theorem. www.math.ucla.edu forwardslash tilde joan

Kripke's "Free choice sequences: A temporal interpretation compatible with acceptance of classical mathematics" (Indag. Math., 2019) inspired e.g. my "Intuitionism at the end of time" (Bull. Symb. Logic, 2019) and "Divergent potentialism: A modal analysis with an application to choice sequences" (Philosophia Mathematica, 2022) by Brauer, Linnebo and Shapiro.

Some Additional References

A. S. Troelstra's Choice Sequences: A Chapter of Intuitionistic Mathematics (Oxford Logic Guides, 1977) describes interesting varieties of choice sequences like the hesitant sequences, which start out free but may (or may not) eventually become lawlike. There are philosophical arguments (e.g. for the Extension Principle) and examples of the practical uses of spreads (particularly finitary spreads, or "fans") in intuitionistic analysis.

Troelstra and van Dalen's two-volume Constructivism in Mathematics: An Introduction (North-Holland 1988) is the most recent comprehensive reference. A list of corrections is posted.

Wim Veldman's "Intuitionism: An Inspiration?" (2021) is a beautifully written, open access (!) introduction by possibly the best intuitionistic mathematician alive today.

Thank you.