Varieties of Potentialism

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Infinity and Intensionality

Classical potentialism

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Actualist

Yes, the natural numbers 0, 1, 2 and so on, are infinite, and they form an *actually* infinite set, a completed collection, which can be used in further mathematical constructions.

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Potentialist may regard the infinite division as incoherent.

After all, if every division is further subdivided, the division itself evaporates into the mist of the limit of an actual infinity of subdivisions.

Infinite lines

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But this is potentialist, since…

The actualist takes the line as already fully extended.

Exhaustion

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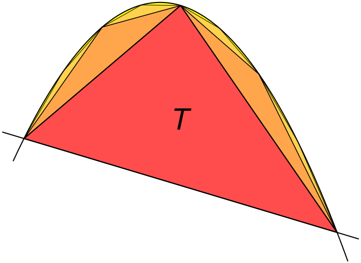

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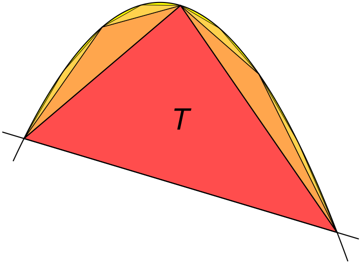

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Archimedes does not use the completed exhaustion, but rather undertakes a double reductio.

Infinitude of primes

Euclid's classic proof is potentialist.

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For any finite list of primes

$$p_1, p_2, \ldots, p_n$$

multiply together and add 1.

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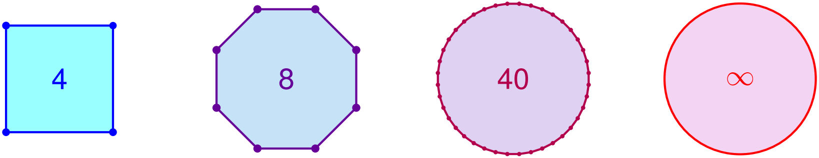
So every finite set of primes can be extended.

Galileo resists the potentialist orthodoxy



Salviati. *If now the change which takes place when you bend a line at angles so as to form now a square, now an octagon, now a polygon of forty, a hundred or a thousand angles, is sufficient to bring into actuality the four, eight, forty, hundred, and thousand parts which, according to you, existed at first only potentially in the straight line, may I not say, with equal right, that, when I have bent the straight line into a polygon having an infinite number of sides, i. e., into a circle, I have reduced to actuality that infinite number of parts which you claimed, while it was straight, were contained in it only potentially? [Gal14, p.47–48]*

Potentialism committed to actual infinity

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Namely, if we can always have more and more, as much as we like, then there is an actual infinity of *possibilities*.

If we really do have all those possibilities, then there is an actually infinite collection.

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It is a kind of illusion to speak of such numbers.

We often find ourselves unable to answer basic questions about them.

For example, is $e^{e^{e^{79}}}$ an integer? (Skewe's number)

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The claim that any particular number is largest seems at best contingent, since we can imagine a number system with one more number.

Ultrafinitists on very large numbers

Harvey Friedman raised the “draw the line” objection with ultrafinitist Yessenin Volpin, concerning existence of

$$2^1, 2^2, 2^3, \ldots, 2^{100}$$

> *I then proceeded to start with 2^1 and asked him whether this is “real” or something to that effect. He virtually immediately said yes. Then I asked about 2^2, and he again said yes, but with a perceptible delay. Then 2^3, and yes, but with more delay. This continued for a couple of more times, till it was obvious how he was handling this objection. Sure, he was prepared to always answer yes, but he was going to take 2^{100} times as long to answer yes to 2^{100} then he would to answering 2^1. There is no way that I could get very far with this.*
>
> *H. Friedman [Fri02, p. 4–5]*

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Thus, again it appears ultrafinitists must weaken induction.

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But not the most radical claims, since 2^{100} exists in every model of $I\Delta_0$.

Lack of a formal theory

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We seem to need already to know what ultrafinitism is, in order to say what it is.

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A theory of finite arithmetic, with a largest number.

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Basic idea: use base-m representation with 4 digits (or any fixed number), where $m = \lfloor\sqrt{N}\rfloor$. Can add and multiply using arithmetic up to N using schoolchild algorithms.

By iterating, one can extend iteratively to a model of $I\Delta_0$.

Connection between Finite arithmetic and $I\Delta_0$

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Every model of finite arithmetic arises by chopping a model of $I\Delta_0$ at a number N and conversely.

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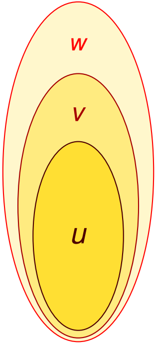
This way of thinking is an entry to the model-theoretic modal perspective on potentialism.

Potentialism via realms of feasibility

"You can have more and more. . . "



Consider the realms that are possible: a Kripke model of possible worlds.

A modal perspective on potentialism

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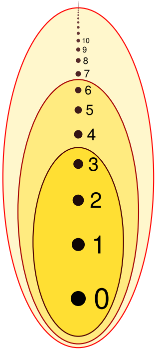
The various universe fragments can be seen as possible worlds in a potentialist system, giving rise to the modal vocabulary:

- $\Diamond\varphi$, if φ holds in some larger world
- $\Box\varphi$, if φ holds in all larger worlds

Initial-segment potentialism



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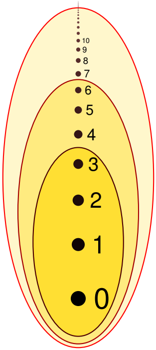

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What are the modal validities?

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Infinitude of primes

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Necessarily, for every number possibly there is a prime above it:

$$\Box\forall x\ \Diamond\exists p\ (x < p \text{ and } p \text{ is prime})$$

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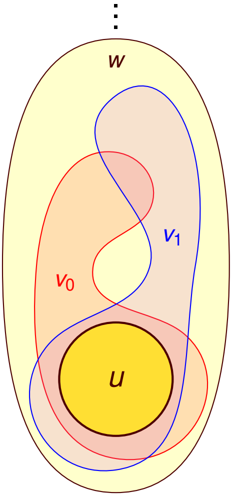
Consider potentialism where numbers arrive in order of complexity of their descriptions. Kolmogorov complexity.

Arbitrary set potentialism



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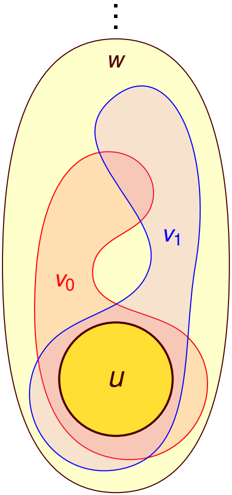

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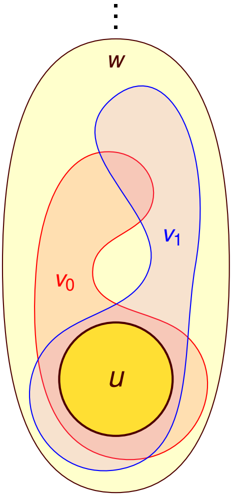

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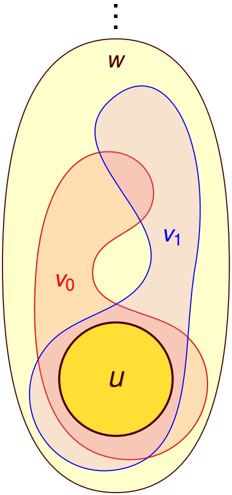

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This variety of potentialism exhibits different modal validities.

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Many different contexts of potentialism.

Potentialist systems

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Alternative approach: $U \sqsubseteq W$ means there is an embedding $U \hookrightarrow W$. A system of counterparts.

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Necessity: φ is necessary over M

$M \models \Box\varphi$ if all such N have $N \models \varphi$.

With this modal language, one can often express sweeping general principles describing how truth varies and propagates through the models as one moves upward in the system.

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Latter case leads to distinction between convergence & amalgamation

In my view, we would benefit from greater philosophical analysis of this distinction for potentialism.

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Good news [HW20]: modal assertions in $\mathrm{Mod}(T)$ are the same!

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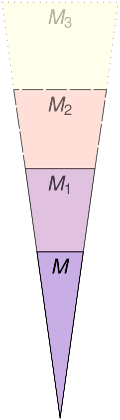
Connected with counterpart theory of individuals.

Feature: algebraic systems often have robust amalgamation.

Good news [HW20]: modal assertions in $\mathrm{Mod}(T)$ are the same!

In fact [AD22]: the two Kripke structures are bisimilar.

Linear inevitabilism



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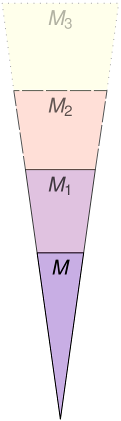

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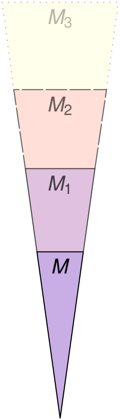

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But also:

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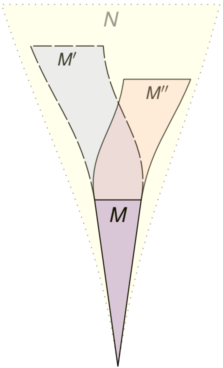
S4.3 is valid.

Convergent potentialism



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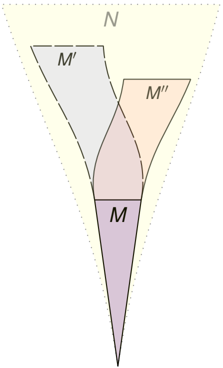

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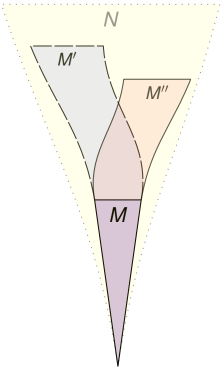

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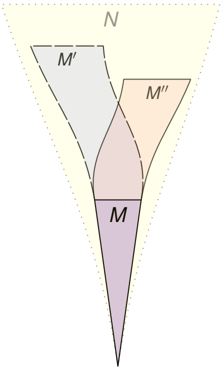

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Only S4.2 is valid.

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Thus, actualist truth reduces to potentialist truth.

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The potentialist denies the limit model exists, yet seems nevertheless to know everything about it.

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My way of saying it: convergent potentialism is implicitly actualist.

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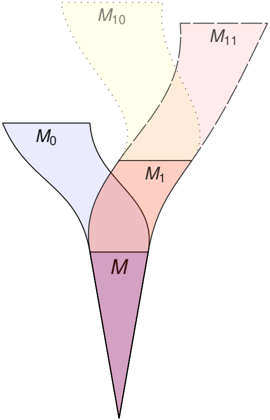
Meanwhile, other more radical forms of potentialism do not have this feature.

Radical branching potentialism



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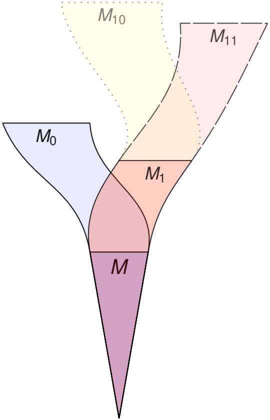

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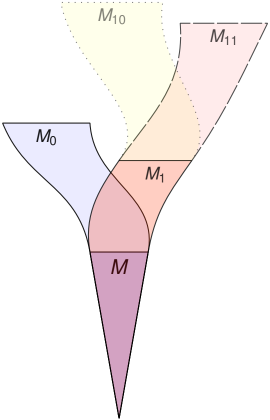

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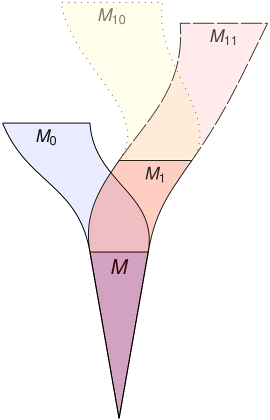

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If a computation is revealed to have output 0, it will never subsequently have output 1, even if that had been possible before.

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The proof makes use the universal algorithm.

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Both theorems are proved with set-theoretic versions of the universal algorithm: the universal definition.

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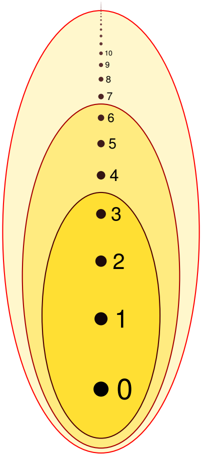
That picture is fundamentally and inherently actualist.

Implicit actualism



This picture represents an understanding of potentialism for the natural numbers, which to my way of thinking arises fundamentally from an actualist conception of what numbers are.

- The coherency of the model arises directly from the actualist conception of $\mathbb{N}$
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In this sense, the view is implicitly actualist.

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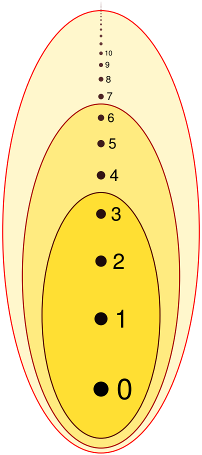

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Similar objections apply to convergent set-theoretic potentialism and other forms of convergent potentialism.

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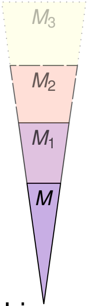
To deny radical branching based on a view that potentialist worlds should converge seems to appeal to an actualist conception.

Varieties of potentialism



Linear
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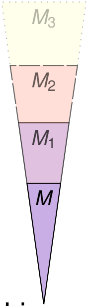
S4.3

Varieties of potentialism



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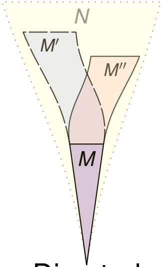

Directed convergence
S4.2

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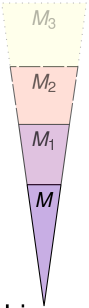

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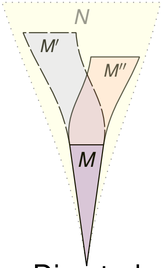

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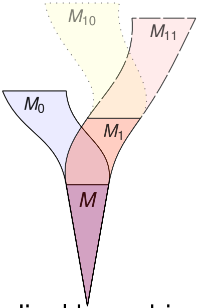

Radical branching possibility
S4

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Thank you.

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