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HUGH MACCOLL AND THE GERMAN ALGEBRA OF LOGIC

In this paper the early reception of Hugh MacColl's logical system up to the 1890s, by the German algebraist of logic, Ernst Schröder, is investigated. In his monumental *Vorlesungen über die Algebra der Logik*, Schröder refers to MacColl as one of his most important precursors. It will be shown that MacColl was a respected member of the logical community of his time, taking his position in the competition for the best (most effective) logical system. Schröder's comparison of the procedures for solving logical problems provided by different logical systems, in particular the different ways of solving Boole's famous "Example 5", is discussed. This discussion will demonstrate the importance of the organon aspect of symbolic logic. Finally some conclusions are drawn concerning later neglect of MacColl's logic.

There is no doubt that Hugh MacColl (1837–1909) was incapable of pioneering a specific tradition in logic. Today, however, some of the particulars of his "Calculus of Equivalent Statements" are regarded as ingenious anticipations of innovations which were only much later introduced to logic. Thus, for a considerable time he was not counted among the important pioneers of symbolic logic. It is, however, hasty to infer from the lack of tradition that MacColl's logical work was ignored by his contemporaries. This can be shown by an investigation of the reception of this work in the German algebra of logic, which is represented (almost exclusively) by Ernst Schröder (1841–1902). His monumental *Vorlesungen über die Algebra der Logik*[1] shows that MacColl was a respected member of the international community of logicians in the late 19th and the early 20th century. His logic took part in the competition of logical systems. In this competition the conceptions of

[1]This work was published in four parts, of which one appeared posthumously: Schröder 1890, 1891, 1895, and 1905, these parts reprinted as Schröder 1966.

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the German mathematicians Ernst Schröder and Gottlob Frege (1848–1925), the American logicians Charles S. Peirce (1839–1914) and Christine Ladd-Franklin (1847–1930), the British, William Stanley Jevons (1835–1882) and John Venn (1834–1923), the Polish Russian Platon Sergeevich Poretskii (1846–1907), and others, concurred. The main task was to solve logical problems, i.e. to schematize them with the help of logical formulas and dissolve them to the unknowns. How the solution was found within the different systems facilitated comparative evaluation of the calculi with respect to their efficiency, elegance and economy. It is obvious that the results of such a comparison could affect the further development of a calculus.

In this paper I will deal with this competition concerning the best logical system from the perspective of the German algebra of logic. How MacColl's logic was received may reveal clues as to why his system never found the recognition it deserved, although it was highly, but critically, esteemed by his contemporaries.[2]

1. Friendly Contests

The competition of logical systems concerned not their theoretical perfection, but practicability. In accordance with the organon conception of rationalistic logic, logical systems were regarded as devices for solving logical and mathematical problems. Logical systems had to be easily applicable. MacColl wrote, for example, in a comment on John Venn's diagrammatic method:

> Where is the formidable array of 6×2^6 (or 384) letters which Mr. Venn, unless I misunderstood his words, supposes the logician obliged to face as a necessary preliminary to all inference in every problem requiring six letters? Whether Dr. Boole's or Prof. Jevons's method can fairly be charged with imposing this heavy labour I am not prepared to say; but my method certainly does not impose it. (MacColl 1880, p. 171)

MacColl's criterion of comparison was the degree of complexity of logical problems which could be mastered by the logical system. The competitors agreed that solving problems was not a simple task. Therefore they quickly looked for help from mechanical devices. In the second

[2]In writing this paper I was able to benefit from preparatory work done by Anthony Christie in the Erlangen research project "Case Studies towards a Social History of Formal Logic", not brought to publication. In particular, it was Christie's idea to interpret the reception of MacColl's writings with respect to the late 19th century competition of logical systems. Cf. the unpublished typescript, Christie 1986. For the research project see Peckhaus 1986, Padilla-Gálvez 1991, and Thiel 1996.

half of the 19th century not only programmable calculating automata, but also logical machines, were developed.[3] With the help of his logical piano, for example, William Stanley Jevons was able to solve mechanically inferences with four terms.[4]

MacColl advocated public competition. In his review of William Stanley Jevons's *Studies in Deductive Logic* (1880), he challenged the author outright:

> Friendly contests are at present being waged in the "Educational Times" among the supporters of rival logical methods; I hope Prof. Jevons will not take it amiss if I venture to invite him to enter the lists with me and there make good the charge of "anti-Boolean confusion" which he brings against my method. (MacColl 1881, p. 43)

MacColl referred to a discussion taking place in the "Mathematical Questions" column of the *Educational Times*, the journal of the British College of Preceptors (cf. Grattan-Guinness 1992). It is astonishing that logical problems found their way into a journal of this type, aimed at a broader audience. But the *Educational Times* was not a singular case. The unusual interest in the new logic in Great Britain after Boole's death led to a great number of contributions, reviews and letters on formal logic in science journals like *Nature* (cf. Christie 1990) and in other national and regional journals.[5]

2. The Reception by Ernst Schröder

From the very beginning the learned competition concerning the best system of logic had an international flavor. The knowledge of the new logical systems in Great Britain was carried into the world primarily through the writings of William Stanley Jevons. His *Principles of Science* (1874) in particular worked as a catalyst. Furthermore, the effect of Alexander Bain's (1818–1903) *Logic* (1870) should not be underestimated. It was devoted to John Stuart Mill's inductive logic, but contained a section on Boole's symbolic logic which stimulated the emergence of research on symbolic logic in Poland.[6] It is safe to assume that even MacColl got to know Boole's algebra of logic via Bain's

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[3] The Analytical Engine of Charles Babbage (1791–1871) is an example, although Babbage was not able to bring it to full operation (cf. Hyman 1982).

[4] Cf. Jevons 1870 and Jevons 1874, pp. 107–114.

[5] Samuel Neil (1825–1901), for example, published in the years 1864 and 1865 a series of articles on "Modern Logicians" in his journal *The British Controversialist and Literary Magazine*, among them comprehensive biographies and reviews on John Stuart Mill (1806–1873, cf. Neil 1864) and George Boole (1815–1864, cf. Neil 1865).

[6] In 1878 Bain's *Logic* was translated into Polish. Cf. Batóg and Murawski 1996.

logic. In the second paper on "The Calculus of Equivalent Statements" (1877/78b), he quoted Boole according to Bain's presentation.[7]

I have shown elsewhere (cf. Peckhaus 1997) that the roots of Schröder's algebra of logic are to be found in the German abstract algebraical conceptions of his time, but not in Boole's algebra of logic. Schröder relied on the general doctrines of forms of Hermann Günther Graßmann (1809–1877, cf. Graßmann 1844) and Hermann Hankel (1839–1873, cf. Hankel 1867), and on Robert Graßmann's (1815–1901) symbolic logic (Graßmann 1872). However, he knew of Boole's calculus and acknowledged its priority from 1874/75 on. Schröder's principal works were the three volumes of his *Vorlesungen über die Algebra der Logik* published between 1890 and 1905. In the first volume (1890), he primarily treated the class calculus after having founded a more general calculus of domains. The second volume, of which a second part appeared posthumously (1891, 1905), was devoted to the propositional calculus. In these two volumes MacColl was, after Charles S. Peirce, the most frequently mentioned author. Peirce's dominance is understandable, insofar as Schröder, in forming his calculi, followed Peirce's model very closely as it was developed in the two papers "On the Algebra of Logic" (1880, 1884). But Schröder always compared Peirce's considerations with the early parts of MacColl's series of papers "The Calculus of Equivalent Statements" published between 1877 and 1880, and he granted MacColl priority when he had anticipated Peirce's results. As far as Schröder was concerned, MacColl's calculus held the status of a preliminary stage of Peirce's algebra of logic.

There are three aspects in Schröder's reception of MacColl's logic which seem to be especially interesting and therefore deserve closer examination:

1. The different efficiencies of the calculi in eliminating terms and resolving logical formulas could provide a measure for the quality of these calculi. Schröder compared his own procedure, derived from Boole's, with Peirce's, the latter being regarded as a natural way of proceeding. He discussed MacColl's method extensively as a preliminary stage.

2. Schröder explicitly accepted MacColl's priority in formulating a propositional logic. In most cases he spoke of a "MacColl-Peircean propositional logic." Nevertheless, he criticized both authors' at-

[7]Elsewhere (MacColl 1878/79, p. 27), he mentions "the kindness of the Rev. Robert Harley for the loan of Boole's 'Laws of Thought'." Referring to Boole's *An Investigation of the Laws of Thought* (1854), he declares that he found many differences, but also numerous points of contact.

tempts to found logic on propositional logic. This conception, he argued, is less general than his own, because he founded the calculus of propositions on the calculi of domains and classes.

3. Schröder stressed the MacColl-Peircean priority of defining material implication, and adopted this definition in his own propositional logic.

2.1. Solution of logical problems

Schröder treated the solution of logical problems in the last two paragraphs of the first volume of his *Vorlesungen über die Algebra der Logik* (1890, §§ 25, 26). He characterized the type of problems discussed in the beginning of § 26 as follows:

> The preceding discussion only concerned problems whose data can be expressed by subsumptions (or equations [...]) between such classes or functions of such in the identical [i.e. Boolean] calculus, and whose solution can also be expressed by propositions of this form. It was important to eliminate certain classes from the data of the problem, to calculate others from these data [...], i.e. to find their subjects and predicates which can be described with the help of the remaining classes. (Schröder 1890, p. 559)

Here Schröder refers to the thirty problems which he solved in the preceding paragraph with the help of his class calculus as far as it was developed at that stage. He then went on to compare his results with solutions provided by alternative calculi. He mentioned Peirce, who had listed in his paper "On the Algebra of Logic" five logical methods in chronological order, by Boole, Jevons, Schröder, MacColl and himself (Peirce 1880, p. 37). Schröder expressed the opinion that these five methods could be reduced to three, because his own was a modified version of Boole's, which thus became obsolete (Schröder 1890, p. 559). Furthermore, the methods of MacColl and Peirce could be combined, because MacColl had paved the way for Peirce's (p. 589).

Schröder chose a problem first published by Boole as a tool for testing the performance of the calculi. It held some prominence among the mathematical and philosophical logicians of the time because of its complexity.[8] Boole's formulation of the problem is quoted in full, but the different solutions are only sketched:[9]

[8] "Example 5" in Boole 1854, pp. 146–149.

[9] Boole 1854, p. 146, cited by Schröder in translation with some revisions (1890, p. 522). This problem was also treated by Hermann Lotze in his "Anmerkung über logischen Calcül" (Lotze 1880, pp. 265–267). Lotze criticized Boole's claim that his solution of the problem shows the advantage of his calculus over syllogistics. Lotze agreed with Boole that it was senseless to try to solve this problem syllo-

Ex. 5. Let the observation of a class of natural productions be supposed to have led to the following general results.

1st, That in whichsoever of these productions the properties A and C are missing, the property E is found, together with one of the properties B and D, but not with both.

2nd, That wherever the properties A and D are found while E is missing, the properties B and C will either both be found or both be missing.

3rd, That wherever the property A is found in conjunction with either B or E, or both of them, there either the property C or the property D will be found, but not both of them. And conversely, wherever the property C or D is found singly, there the property A will be found in conjunction with either B or E, or both of them.

Let it then be required to ascertain, first, what in any particular instance may be concluded from the ascertained presence of the property A, with reference to the properties B, C, and D; also whether any relations exist independently among the properties B, C, and D. Secondly, what may be concluded in like manner respecting the property B, and the properties A, C, and D. (Boole 1854, p. 146)

In his translation Schröder labelled the data α, β, and γ, and he split the two questions into four:

Let it be required to ascertain,

first, what in any particular instance may be concluded from the ascertained presence of the property A, with reference to the properties B, C, and D,

secondly, also to decide whether any relations exist independently from the presence or absence of the other properties among the presence or absence of the properties B, C, and D (and, if yes, which?),

thirdly, what may be concluded in like manner from the existence of the property B with respect to the properties A, C, and D (and vice versa, when the existence or absence of the property B can be inferred from that of the properties of the latter group),

fourthly, to state what follows for the properties A, C, D as such. (Schröder 1890, p. 522)

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gistically, but did not regard the calculatory procedure as obvious. He preferred a combinatorial way which he obviously adopted from Jevons. This combinatorial way "presents itself automatically as the more appropriate" (Lotze 1880, p. 266). Jevons's combinatorial procedure was a subject of correspondence between Lotze and Schröder. Schröder reported on this correspondence, criticizing Lotze's devaluation of the calculatory method (cf. Schröder 1890, pp. 566–568). Gottlob Frege, like Lotze, criticized the artificiality of this problem in his comparison of the *Begriffsschrift* with the Boolean calculus (Frege 1983, p. 52). Nevertheless, he also tried to solve the problem. Frege's pathbreaking solution is thoroughly discussed by Peter Schroeder-Heister (1997). A favorable treatment of this problem can be found in Wilhelm Wundt's logic (1880, p. 357). Gottfried Gabriel suggested in 1989 the examination of the different solutions in order to obtain criteria for a comparison of different systems of logic, i.e. traditional logic, algebra of logic, and Frege's *Begriffsschrift*(Gabriel 1989, p. XXIII).

Schröder's notations will be used to sketch out his solution. Lowercase Latin letters stand for classes; their properties are marked with respective capitals. Logical addition (adjunction) is marked by +, logical multiplication (conjunction) by juxtaposition. Negation is symbolized by an appended negation stroke, e.g. $a_\shortmid$ stands for the negation of a. The subsumption symbol $\Subset$ indicates class inclusion in the class calculus. The class symbols 0 and 1 stand for the empty class and the universal class, respectively. The functional symbol $f(x)$ represents in the identical (i.e. Boolean) calculus a complex expression containing x (or $x_\shortmid$) and other symbols connected with the help of basic logical operations, identical multiplication, addition and negation (cf. Schröder 1890, p. 401). Schröder's solution will be outlined in as far as it is necessary to compare it with the alternative solutions discussed.

In an initial step Schröder presented the data, i.e. the conditions α–γ, as subsumptions or equations. In Schröder's calculus, equality is derived from subsumption: $a = b$ stands for a subsumption relation between the terms a ("subject") and b ("predicate") which is valid in both directions at the same time. $a = b$ is thus defined as $(a \Subset b)(b \Subset a)$. In Schröder's symbolism the data α–γ can be formalized as follows:

$$\begin{array}{lrcl} \alpha: & a_\shortmid c_\shortmid & \Subset & (bd_\shortmid + b_\shortmid d)e \\ \beta: & ade_\shortmid & \Subset & bc + b_\shortmid c_\shortmid \\ \gamma: & a(b+e) & = & cd_\shortmid + c_\shortmid d\,. \end{array} \tag{1}$$

These formulas contain the class symbol e as related to the property E, which then has to be eliminated because it does not affect the solutions of the questions. In order to eliminate this class symbol, Schröder put the equations (1) to 0 on the right hand side, and finally combined these three equations by conjunction into one. For this purpose he could use two theorems proven earlier:[10]

$38_\times$ $$(a \Subset b) = (ab_\shortmid = 0)$$

and

$39_\times$ $$(a = b) = (ab_\shortmid + a_\shortmid b = 0).$$

After combining the modified premises the following equation results:

$$a_\shortmid c_\shortmid(bd + b_\shortmid d_\shortmid + e_\shortmid) + ade_\shortmid(bc_\shortmid + b_\shortmid c) + \\ +a(b+e)(cd + c_\shortmid d_\shortmid) + (a_\shortmid + b_\shortmid e_\shortmid)(cd_\shortmid + c_\shortmid d) = 0\,.$$

[10] Schröder erroneously mentions Theorem 39_+ instead of $39_\times$ which allows, however, an equation to be brought to 1.

Several steps of calculation are required for the elimination of e. They result in the formula

$$a(cd + bc_{\shortmid}d_{\shortmid}) + a_{\shortmid}(cd_{\shortmid} + c_{\shortmid}d + b_{\shortmid}c_{\shortmid}d_{\shortmid}) = 0\,.$$

This formula is the starting point for further eliminations and resolutions of certain class symbols, finally leading to an answer for the questions.

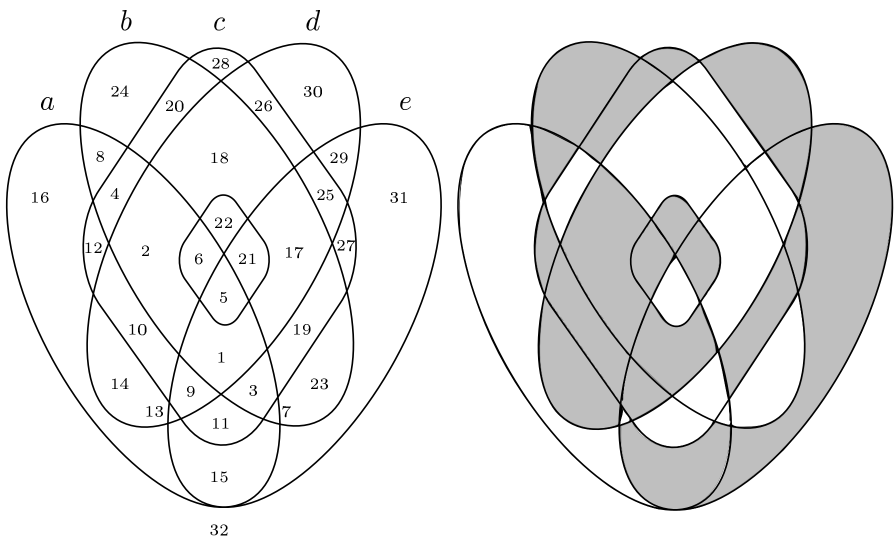
Schröder stressed the similarity of his method with that of Boole, which he considered, however, to be "definitely settled" because of his modifications. As far as Schröder was concerned, Boole's method was therefore only of historical interest. Its disadvantages resulted from the lack of a sign for negation—Boole had to write $1 - x$ for $x_{\shortmid}$—and from the interpretation of the logical "or" as an exclusive "or". The inadequacy of Boole's language led to logically uninterpretable expressions in the course of calculating logical equations according to the model of arithmetic.

Schröder started his discussion of alternatives with Jevons's method which he called "without art" ("kunstlos"), although it was the "nearest at hand or most unsophisticated". Jevons proposed this "Crossing-off procedure" ("Ausmusterungsverfahren") in his *Pure Logic* (1864). According to Schröder, it consisted in

> writing down for all classes mentioned in the formulation of the problem all the possible cases which can be thought of with respect to the presence or absence of one in relation to another, then crossing off all cases which are excluded from the thinkable combinations by the data of the problem as inadmissable, and trying to pick out the answers to the questions posed by the problem from the remaining ones. (Schröder 1890, p. 560)

Schröder applied Jevons's method to Boole's problem (1890, pp. 562–566). It contained five class symbols; therefore $2^5 = 32$ combinations had to be considered, of which eleven are valid. Schröder criticized the complexity of the combinatorial method, which grows with the square of the number of class symbols occurring. He furthermore claimed that the procedure is not really calculatory, but that it is based instead on a "mental comparison" of combinations and premises (pp. 567–568).

Schröder also discussed the graphical extension of this method presented by John Venn in his *Symbolic Logic* of 1881 (Schröder 1890, pp. 569–573). Venn symbolized the relations between the extensions of classes if two or three class symbols are involved by circles, by ellipses with four class symbols, and by ellipses together with a ring in the form of a rhombus with five class symbols. The procedure is similar to Jevons's crossing-off method, because the fields not present according to the data of the problem are erased by hatching them. With respect to the complexity of the problems which can be treated, Venn's procedure was even more restricted than Jevons's, because the schemes for symbols of more than five classes become rather intricate.[11] Schröder admitted that Venn's method had the advantage that every logical problem which could be presented in an intuitive form could be solved as soon as it was symbolized with the help of the graphical scheme. The scheme proposed by Venn for five class-symbols is shown on the left hand side of the figure below. On the right hand side the solution of the Boolean problem is given.[12] The exterior field 32 should be hatched as well. This has not been done for the sake of descriptiveness.[13]



[11]Schröder 1890, p. 569. Today, however, graphical procedures have been developed with which greater complexities can be handled.

[12]Schröder used the following algorithm for numbering the fields of the Venn diagram.

	a	b	c	d	e
1	1	1	1	1	1
2	1	1	1	1	0
3	1	1	1	0	1
4	1	1	1	0	0
5	1	1	0	1	1
6	1	1	0	1	0
7	1	1	0	0	1
8	1	1	0	0	0
			⋮		
32	0	0	0	0	0

I thank Peter Bernhard (Erlangen) for sharing this information.

While Schröder was critical of the methods of Jevons and Venn, he praised those of MacColl and Peirce for being equal to his own in their efficiency (1890, p. 560). He illustrated the relation between the different methods with the following metaphor: While he himself ties the different skeins of premises into a single bale (i.e. the united equation) and then hacks his way through it, Peirce separates each skein into thin threads and cuts them individually or binds them together again if necessary. Jevons, on the other hand, would make chaff and banter of the whole thing.[14] If one modifies Peirce's procedure so that the clues are separated only as far as is needed to isolate the symbols for eliminations and the unknowns, it will come close to MacColl's. Schröder acknowledged that the variants of MacColl and Peirce were natural and simple, but criticized their lengthiness (1890, p. 573).

Schröder reconstructed Peirce's method as a sequence of six steps ("Prozesse") (1890, pp. 574–584). He followed Peirce's own presentation (cf. 1880, pp. 37–42).

1. In an initial step the premises are expressed as subsumptions.

2. Then every subject (the term on the left hand side of a subsumption) is developed as a sum, every predicate (the term on the right hand side) as a product, using the schemes

$$\begin{array}{llcll} 44_+ & f(x) & = & f(1)x + f(0)x_{\shortmid} & \text{and} \\ 44_\times & f(x) & = & \{f(0)+x\}\{f(1)+x_{\shortmid}\}\,. & \end{array}$$

3. In the third step all complex subsumptions are reduced, e.g.

$$s + s' + s'' + \ldots \Subset pp'p'' \ldots$$

into the subsumptions

$$\begin{array}{l} s \Subset p, s \Subset p', s \Subset p'', \ldots \quad , \\ s' \Subset p, s' \Subset p', s' \Subset p'', \ldots \quad , \\ s'' \Subset p, s'' \Subset p', s'' \Subset p'', \ldots \quad , \\ \vdots \end{array}$$

[13] Schröder corrects the solution given by Venn (Venn 1881, p. 281) by hatching field 24. Venn acknowledged this correction in the second edition of his *Symbolic Logic* (1894, p. 352, n. 1).

[14] "Bei dieser wurden die verschiedenen Knäuel der Prämissen oder Data des Problems erst fest zu einem einzigen Knoten geschürzt (der vereinigten Gleichung) und dieser dann durchhauen (bei der Elimination).

Beim Peirce'schen Verfahren aber werden jene Knäuel in ihre dünnsten Fäden auseinandergeleft und die erforderlichen einzeln zerschnitten (oder auch neu nach Bedarf verknüpft)—wogegen die Jevons'sche Methode sogleich ein Häcksel aus dem Ganzen machte!" (Schröder 1890, p. 573)

4. The fourth step is devoted to the necessary eliminations.
5. In the fifth step all the terms, where the unknown x can be found at subject or predicate position, are picked up, and finally ...
6. ... united in the last step. With the help of the resulting formula the unknown can be calculated.

Schröder saw the advantages of Peirce's method over Boole's in the fact that it operates with subsumptions and not with equations, and that it preserves the subject-predicate structure, which "thoroughly matches the judging functions of ordinary reasoning" (1890, p. 584)—but these are advantages in Schröder's method as well. Peirce's method has the further advantage that it is not necessary to bring the equations to zero on the right hand side and then to unite them to one single equation.

Schröder closed his considerations on the class calculus (1890, pp. 589–592) with a discussion of MacColl's method. He stated that MacColl invented this method independently, but nevertheless rather belatedly, in order to solve the problems of the Boolean calculus. It differed, however, not as much from the modified Boolean method (i.e., Schröder's method) as MacColl himself thought. Schröder stressed that he agreed with Venn, who had written in his assessment (Venn 1881, p. 37; 1894, p. 492) that MacColl's symbolical method is "practically identical with those of Peirce and Schröder."

This assessment is accurate if one regards MacColl's own evaluation of the differences between his system and those of Boole and Jevons. In the third part of the series of papers on "The Calculus of Equivalent Statements", he gave a list of the points of difference:

> 1. With me every single letter, as well as every combination of letters, always denotes a *statement*.
> 2. I use a symbol (the symbol :) to denote that the statement following it is true provided the statement preceding it be true.
> 3. I use a special symbol—namely, an accent—to express denial; and this accent, like the minus sign in ordinary algebra, may be made to affect a multinomial statement of any complexity. (MacColl 1878/79, p. 27)

Relating implication and subsumption, the latter being the class logical equivalent of the former, MacColl presented important modifications to the calculi of Boole and Jevons which were later also introduced by Schröder. It is noteworthy that MacColl did not mention his use of the inclusive "or". The reason may be that MacColl, in the context of the quoted passage, paralleled the calculi of Boole and Jevons, using

Jevons's *Pure Logic* (1864), where the exclusive "or" had already been replaced by the inclusive "or".

It is a matter of course that Schröder recognized that MacColl's formulas emerged from the propositional calculus, the "calculus of equivalent statements", in which the symbols 0 and 1 were not class symbols but interpreted as truth values. Schröder discussed this method in his more general class logic because MacColl also treated the Boolean class logical example 5 (cf. MacColl 1878/79, pp. 23–25).

According to Schröder's analysis, MacColl's solution was based on the two equations named "rule 22" (cf. MacColl 1878/79, p. 19):

$$xf(x) = xf(1);\ x'f(x) = x'f(0)\,.$$

Schröder had dicussed these equations as "theorems of MacColl's" at an earlier stage in his *Vorlesungen* (Schröder 1890, § 19, p. 420).[15]

Schröder regarded it as an advantage of MacColl's procedure that the premises were not united. In this point, he stated, MacColl was a precursor of Peirce. But he denied further advantages over his or Peirce's method, for example with respect to printing economy, a better survey, or more comfort in working (1890, pp. 591–592).

It is curious that Schröder returned to MacColl's method in the second volume of his *Vorlesungen*, which is devoted to the calculus of propositions. There he retracted his assessment that MacColl's solution was not really original (1891, p. 391). Schröder then stated that MacColl's method, as presented above, was only a scheme. In fact, Schröder said, MacColl had used another method, which was indeed original and advantageous. Schröder sketched it as follows (1891, pp. 304–305):

In a given product of propositions $F(x, y)$, y is to be eliminated and x is to be calculated. The following four implications are used

$$\begin{array}{lcl|lcl} xy & ⋹ & F(1,1) & x_{|}y & ⋹ & F(0,1) \\ xy_{|} & ⋹ & F(1,0) & x_{|}y_{|} & ⋹ & F(0,0) \end{array}$$

By addition and using the theorems $x = xy + xy_{|}$ and $x = x_{|}y + x_{|}y_{|}$, y can be eliminated, resulting in

$$x ⋹ F(1,1) + F(1,0) \mid x_{|} ⋹ F(0,1) + F(0,0)\,.$$

By contraposition the solutions can be given

$$F_{|}(1,1)F_{|}(1,0) ⋹ x_{|} \mid F_{|}(0,1)F_{|}(0,0) ⋹ x\,.$$

[15] In MacColl's notation the apostrophe denotes negation. MacColl himself said that he used the implications $xf(x) : f(1)$, $x'f(x) : f(0)$ named as rule 23.

2.2. Calculus of propositions

The fact is often overlooked that Schröder, in the second volume of his *Vorlesungen*, presented a highly elaborate propositional and predicate logic, which was not motivated by Frege. He adopted the quantifiers of Peirce and Oscar Howard Mitchell (1851–1889), using a sum and product notation. In the hierarchy of Schröderian logic, propositional logic was only in third position. Schröder started with a calculus of domains, consisting of manifolds of elements. He spoke of a class calculus only if these elements could be individuated and combined to form classes. It was a further specialization if the calculus concerned propositions (judgements or "statements"). With this architecture in mind, it is reasonable for Schröder to reproach MacColl and Peirce for putting the cart before the horse in founding the logical calculus on the calculus of propositions. His hierarchy, Schröder stressed, had the advantage of being more general, and it was also better in didactical respects, because it was not necessary to have the complete syllogistical apparatus at hand from the beginning. He compared the difficulties of reading Peirce's papers on logic (his argument was, of course, also valid for MacColl's papers) with the difficulties of a student, who "should learn a language which is unknown for him from a grammar which is written in the same language" (1891, p. 276).

2.3. Material implication

In the algebraic view, formulas and symbols in logic are interpreted in different ways, depending on the different contexts of application. The subsumption, which is interpreted in the class calculus as an "integrative operation" ("Einordnung") leading to the equality or subordination of classes, becomes an implication in the propositional calculus. In the identical calculus Schröder defined subsumption as material implication

$$(A \preccurlyeq B) = (A_{|} + B)\,,$$

and interpreted the formula as follows: "The 'validity class' of the subsumption $A \preccurlyeq B$ is the class of occasions, during which A is not valid or B is valid" (Schröder 1891, p. 69). But he differed from Peirce before him, in that he obtained the equation by identical transformations:

$$(A \preccurlyeq B) = (AB_{|} = 0) = \{(AB_{|})_{|} = \dot{1}\} = (A_{|} + B = \dot{1}) = A_{|} + B\,.$$

The dotted 1 stands for the truth value "true". Schröder used the term $A = \dot{1}$ more exactly to express that the proposition A is always, at any time, and on all occasions, valid. Marking the 1 with a dot is necessary to distinguish it from the arithmetical 1, which occurs in quantifications over a numerical index.

Schröder granted the priority of material implication to MacColl and Peirce. It is astonishing that MacColl gave a far more modern interpretation of the formulas, ten years before Schröder. In his interpretation, $A = 1$ means that the statement A is true; $A = 0$ that it is false (1877/78a, p. 9). The expression $A : B$ means that statement A implies statement B, i.e. in any case if A is true, then B is true (1877/78b, p. 177).

I would finally like to suggest that MacColl didn't grasp the algebraic approach of Schröder and Peirce. This became obvious when he criticized the opinion that Peirce's operation of illation and Schröder's subsumption were equivalent to his implication, whereas, according to his opinion, these operations only denote class inclusion (MacColl 1906, § 74). He did not realize that in the Peirce-Schröderian algebra of logic, symbols for operations and for relations between variables were only denoted in a schematic way. The meaning of the symbols thus depended on the meaning of the variables.

3. Conclusion

As far as Schröder was concerned, MacColl appeared to be a successor of Boole's in his algebra of logic who, unlike Jevons, kept a logical notation closely analogous to mathematics, but who, like Jevons, was also able to avoid the limitations of Boole's logic. In combining Boolean Algebra (which was founded by Jevons) with mathematical symbolism, he became a precursor of Peirce. Schröder appears to have heard of MacColl via Peirce. But throughout his work he acknowledged MacColl's priority. His comments sometimes read as if MacColl's calculus is a purely historical precursor of Peirce's logic. Schröder was quite attracted to Peirce's ideas, possibly because his interests in logic changed while he was still writing the first two volumes of his *Vorlesungen*. His new interest was an "Algebra and Logic of Relatives", the first volume appearing in 1895, which he elaborated according to Peirce's model. Therefore Schröder's presentation of MacColl's system had no crucial effect on the reception of MacColl's ideas.

This state of affairs was not changed by the fact that MacColl cut a good figure in the competition of logical systems during the heyday of the algebra of logic between 1864 and 1890, at least according to Schröder. The modal operations, characteristic for MacColl' s *Symbolic Logic* of 1906, were not formulated in the early parts of the series of papers "The Calculus of Equivalent Statements" to which Schröder referred. Nevertheless, it is possible to draw some conclusions from the early reception, compared to the acceptance of MacColl's later work. The logic discussion of the time shows the importance of the organon aspect of logic. The logical calculus was regarded as a tool for the solution of logical problems, but also for problems from other areas which could be translated into logical language. These areas included mathematics, the philosophy of science, jurisprudence, but also genealogy. Questions of symbolism played a predominant role in evaluating the usability of the calculi. The bulky symbolism of MacColl's later work could only prevent broader acceptance. Thus, MacColl shared the same fate as Frege.

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