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MACCOLL ON MODALITIES

This paper tries to reconstruct modal principles advanced by MacColl. It is argued that he had the basic rules of the modal square of opposition. On the other hand, his proofs of contradictions stemming from iterating modalities are incorrect.

Frege and Russell, the fathers of mathematical logic, were not very much interested in modalities and relations between them.[1] For Frege:

> The apoidictic judgment differs from the assertory in that it suggests the existence of universal judgements from which the proposition can be inferred, while in the case of the assertory one such suggestion is lacking. By saying that a proposition is necessary I give a hint about the grounds for my judgments. *But, since this does not affect the conceptual content of the judgment, the form of the apoidictic judgment has no significance for us.*
>
> If a proposition is advanced as possible, either the speaker is suspending judgment by suggesting that he knows no laws from which the negation of the proposition would follow or he says that the generalization of this negation is false. In the latter case we have what is usually called a *particular affirmative judgment* ... "It is possible that the earth will at some time collide with another heavenly body" is an instance of the first kind, and "A cold can result in death" of the second. (Frege 1879, p. 13; Frege's italics)

This quotation shows that Frege located modalities outside the domain of pure logic.

Russell 1903 offers no treatment of modalities. Appendix C of *Principia Mathematica*, on truth-functions and other propositional forms, mentions epistemic operators (assertion, belief), but contains nothing about alethic, that is, proper modal propositions. In Russell 1905, we find a form '$C(x)$', where x is a free variable, as a general scheme of a proposition. Further, Russell considers the phrases '$C(x)$ is always

[1] See Rescher 1974 and Dejnožka 1999 on Russell and his objections to MacColl and modal logic.

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true' and '$C(x)$ is sometimes true', which are often (Russell does not mention this) considered as connected with modalities. For Russell, these phrases mean, respectively, 'C(everything)' and 'C(something)'. Thus, we can say that Russell reduced the logical meaning of modalities to quantifiers. This is confirmed by the following passage from his *Introduction to Mathematical Philosophy*, which is probably an allusion to MacColl:

> Another set of notions as to which philosophy has allowed itself to fall into hopeless confusions through not sufficiently separating propositions and propositional functions are the notions of "modality": *necessary, possible,* and *impossible* ... In fact, however, there was never any clear account of what was added to truth by the conception of necessity. In the case of propositional functions, the three-fold division is obvious. If "ϕx" is an undetermined value of a certain propositional function, it will be *necessary* if the function is always true, *possible* if it is sometimes true and *impossible* if it is never true. (Russell 1919, p. 165; Russell's italics)

Not even more traditional logicians were involved in formal studies of modalities. This becomes clear if we inspect the logical treatises of Sigwart, Erdmann and other authors of the turn of the 20th century. Thus, the great tradition in modal logic going back to Aristotle, and successfully continued in the Middle Ages, was almost entirely neglected until the 1930s. Of course, almost every logician considered so-called modal sentences: problematic (expressing possibility) and apoidictic (expressing necessity), but almost everything discussed was limited to analysis of various meanings of modal concepts, and not of formal relations between modal sentences. We can find something in Höfler, who described relations from the square of oppositions for modals in 1917. Even Lewis 1912 contains nothing about the logic of modalities, which started with Lewis 1918, really a pioneering work in the field. Causes of this situation seem to be these. Firstly, analysis of modalities was burdened by very obscure epistemological and psychological considerations. Secondly, the dogma of extensionalism accepted by Frege, Russell and the majority of formal logicians of that time was responsible for the neglect of modal logic.

Hugh MacColl is a notable exception in this respect. He gave an analysis of modalities, established some connections between them and stated some problems. The question must have been important to him, because he considered it in his papers, his letters to Russell and in his main book published in 1906. The treatment in MacColl 1906 is the most extensive and I will use this source. I will try to reconstruct MacColl's ideas concerning modalities using his terminology, but not his symbolism.

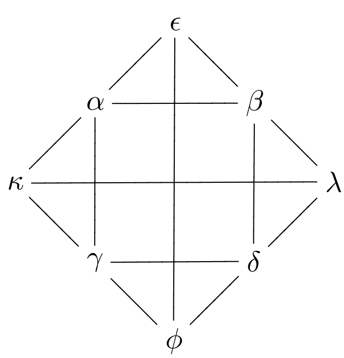
The importance of modalities for MacColl was evidently connected with his understanding of implication (see MacColl 1906, p. 7). Let $T(A)$ mean 'A is true'. Thus, '$T(A)$ implies $T(B)$' means (a) if A belongs to the set of truths, then B belongs to the set of truths, (b) it is impossible that A belongs to the set of truths without B belonging to the set of truths, (c) it is certain that either A does not belong to the set of truths or B belongs to the set of truths. The locutions are not only equivalent for MacColl, they are even synonymous. I do not discuss whether he is right or not. I mention this view of MacColl's just in order to show that modalities were important to him for fundamental reasons, having obvious relations, speaking in a more contemporary manner, with strict implication and many-valueness.

MacColl distinguishes five attributes of statements considered in pure or abstract logic: truth (T), falsity (F), certainty (N), impossibility (I) and variability (C). Let me quote relevant explanations (adopted in my symbolism):

> ... the symbol ... $C(A)$ asserts that A is *variable* (possible, but uncertain). The symbol $T(A)$ only asserts that A is true in a particular case or instance. The symbol $N(A)$ asserts more than this: it asserts that A is *certain*, that A is always true (or true in *every case* within the limits of our data and definition, that its probability is 1). The symbol $F(A)$ only asserts that A is false in a particular case or instance; it says nothing as to the truth or falsehood of A in other instances. The symbol $I(A)$ asserts more than this; it asserts that A contradicts some datum or definition, that its probability is 0. Thus, $T(A)$ and $F(A)$ are simply *assertive*; each refers only to one case, and raises no question as to data and probability. The symbol $C(A)$ (A is variable) is equivalent to $\neg I(A) \wedge \neg N(A)$; it asserts that A is neither *impossible* nor *certain*, that is, that A is *possible* but *uncertain*. In other words, $C(A)$ asserts that the probability of A is neither 0 nor 1, but some proper fraction between the two. (MacColl 1906, pp. 6–7; MacColl's italics)

There are certain interpretative problems concerning probability, or contradicting some datum or definitions which I will not enter into here. However, we can derive from MacColl's explanations clear formal ideas. Let's think about instances or cases as possible worlds, or temporal points. Thus, truth *simpliciter* means truth in some possible world (at a certain temporal point), falsity *simpliciter*—falsity in some possible world (at a certain temporal point), certainty—truth in all possible worlds (at all temporal points), impossibility—falsity in all possible worlds (at all temporal points), and variable—truth in some possible world (at a certain temporal point) and falsity in some possible world (at a certain temporal point). In any case, we are entitled to treat certainties as necessary statements and variables as contingent statements. Further, MacColl notes that $FI(A)$ is not generally equivalent to $IF(A)$, which is of course a correct observation, and extends his definition of implication to other modalities—this point is not relevant for my further considerations. MacColl states (pp. 12–19) the following theorems on modalities: (a) $N(A \vee \neg A)$, (b) $N(T(A) \vee F(A))$, (c) $N(T(A) \wedge F(A))$, (d) $N(N(A) \vee I(A) \vee C(A))$, (e) $N(A) \Rightarrow T(A)$, (f) $I(A) \Rightarrow F(A)$, (g) $N(A) \Leftrightarrow I(\neg A)$, (h) $I(A) \Leftrightarrow N(\neg A)$, (i) $C(A) \Leftrightarrow C(\neg A)$, (j) $\neg C(A) \Leftrightarrow (N(A) \wedge I(A))$.

Thus, (a)–(c) assert (roughly speaking) that modal logic is an extension of classical logic, (d) that every statement is necessary, impossible or contingent, (e) that necessity implies truth, (f) that impossibility implies falsity, (g) and (h) establish the mutual definibility of necessity and impossibility *via* negation, (i) that $C(A)$ and $C(\neg A)$ are equivalent, and (j) that non-contingency is equivalent to necessity or impossibility. There is a problem with the definition of possibility in MacColl. In one place he says that possibility (M) is defined by $M(A) \Leftrightarrow \neg I(A)$. This suggests the standard understanding of $M(A)$ as $\neg N(\neg A)$. If we take this route, MacColl's formal ideas on modalities can be summarized by the following diagram.



We have the following dependencies:

$(a)\ \alpha \Rightarrow \gamma$,	$(g)\ \alpha \Rightarrow \kappa$,	$(m)\ \alpha \Rightarrow \epsilon$,
$(b)\ \neg(\alpha \Leftrightarrow \delta)$,	$(h)\ \beta \Rightarrow \lambda$,	$(n)\ \beta \Rightarrow \epsilon$,
$(c)\ \neg(\beta \Leftrightarrow \gamma)$,	$(i)\ \kappa \Rightarrow \gamma$,	$(o)\ \phi \Rightarrow \gamma$,
$(d)\ \neg(\alpha \wedge \beta)$,	$(j)\ \lambda \Rightarrow \delta$,	$(p)\ \phi \Rightarrow \delta$,
$(e)\ \beta \Rightarrow \delta$,	$(k)\ \epsilon \Leftrightarrow \alpha \vee \beta$,	$(r)\ \neg(\epsilon \Leftrightarrow \phi)$,
$(f)\ \gamma \vee \delta$,	$(l)\ \phi \Leftrightarrow \gamma \wedge \delta$,	$(s)\ \alpha \wedge \beta \wedge \phi$.

Now interpret α as $N(A)$, β as $I(A)$, γ as $M(A)$, δ as $M(\neg A)$, κ as $T(A)$, λ as $F(A)$, and ϕ as $C(A)$. We get MacColl's modal logic as an interpretation of a square of oppositions extended (by adding κ, λ, ϵ and ϕ) for modal sentences. If this interpretation of MacColl is correct, I think that he was the first who was conscious of that in modern times, regardless of his unclarities about possibility. He notes (see MacColl 1906, p. 105) that four modalities of the traditional logic are represented by the formula $N(A) \vee I(A) \vee T(A) \wedge C(A) \vee F(A) \wedge C(A)$. It is a conjunction of (b) and (d) and a theorem. One can guess that $T(A) \vee C(A)$ represents possibility (M) and $F(A) \vee C(A)$ non-necessity ($M\neg A$). However, this hypothesis is inconsistent with the standard definition of $M(A)$ as $\neg N(\neg A)$ and $M(\neg A)$ as $\neg N(A)$. Unfortunately, MacColl does not say exactly which "traditional" modalities he had in his mind. A hint for understanding his views we find in the following explanations (pp. 14–18). Let $P(A)$ mean 'A is probable' (the likelihood of A is greater than 1/2), $Q(A)$ means 'A is improbable' (the likelihood of A is less than 1/2), and $U(A)$ means 'A is uncertain'; other modals have their already explicated meanings. Now MacColl stipulates: (a) the denial of truth is an untruth, and, conversely, (b) the denial of probability is an improbability, and, conversely, (c) the denial of certainty is an impossibility, and, conversely, (d) the denial of variable is a variable, and (e) the denial of possibility is uncertainty, and conversely. The stipulation (a) is obvious (MacColl obviously identifies here untruth and falsity), (b) is unclear, because we do not know whether the probable includes certain or not, (d) is obvious, but (c) and (e) contradict the standard understanding of modalities. MacColl explains why the denial of possibility is uncertainty and not impossibility. Consider, he says, the statement (i) 'It will rain tomorrow'. Now the statement (ii) 'It will not rain tomorrow' is its denial. The statement (i) is a possibility and (ii) merely uncertain, not an impossibility. In particular, in order to prove that a denial of a possibility is an uncertainty we have to prove that the possibility in question implies the uncertainty of this possibility.

The problem with MacColl's explanations is connected with the fact that he passed from an analysis of modalized statement to the status of unmodalized ones. His example expresses a typical future contingency ('It will rain tomorrow'). It is fairly obvious that here MacColl confused possibility and contingency, because he constructed his example as expressing possibility and non-necessity (possibility not), that is, just contingency. Moreover, he also confused denials of modalized statements with denials of arguments of modal operators in the situation in which their modal status is determined; for example, the denial of $M(A)$ with the denial of A itself, provided that we know that A is a possibility. Assuming our diagram, a proper analysis of 'it will rain tomorrow' is that it is located at the point ϵ. Thus, the denial of $C(A)$ is $N(A) \vee I(A)$, but if we know that A is a contingency, $\neg A$ is also a contingency. Hence, MacColl should say that if A is uncertainty and possibility, its denial is the same. Of course, he is right that in order to prove that A is a contingency, we must prove that this fact implies that $\neg A$ is uncertainty, but without further ado his explanations are burdened by an ambiguity of 'possible' and 'uncertain'. It is not blocked by a remark (MacColl 1906, p. 15, footnote) that we should understand the denial of certainty as a denial of a certain statement. Of course, the denial of a tautology (a certainty) is a contradiction (an impossibility), which legitimizes (c), but it also leads to ambiguity. It is also possible that this second treatment of modalities is more consistent with many-valued logic than with extensions of classical logic.

MacColl constructs an antinomy concerning so-called second-degree modal statements. A statement of the form $S(A)$, where S expresses a modality, is called a first-degree modal statement. Now a statement $SS(A)$ is second-degree, a statement $SSS(A)$ third-degree, and so on. Take a statement $CC(A)$. We can assume that any statement is a certainty, an impossibility or a variable. Assume that A is a certainty. This means that A belongs to the set of certainties. On the other hand, provided that A is a certainty, $C(A)$ means that A is a variable (contingency). Thus, we arrive at a conclusion that a certainty is a variable which is impossible. So $IC(A)$. But, in this situation $CC(A)$ means that an impossibility is a variable; in the terminology of this paper a contingency is an impossibility. This is a contradiction. Similarly, we prove that if is A is an impossibility, $CC(A)$ is an impossibility too. Thus, a variable is an impossibility. Finally, assume that A is a variable. In this situation the formula $C(A)$ is self-evidently true and certain. But the formula $CC(A)$ asserts that a certainty is a variable, which leads to a contradiction, that is, an impossibility. On the other hand, take any set of arbitrary statements which consists of certainties, impossibilities and variables. We can check the probability that a statement A taken from this set at random is a certainty, a variable or an impossibility. Thus, the sentences $N(A)$, $C(A)$ and $I(A)$ are variables. Then, $CC(A)$ is always true.

MacColl solves the problem in the following manner:

After some reflexion, I found that the second of these antinomies (namely that $CC(A)$ is *not* self-contradictory) is the true one. Where then is the error in the first argument? It consists in this, that it tacitly assumes that A *must* either be *permanently* a certainty, or *permanently* an impossibility, or *permanently* a variable—an assumption for which there is no warrant. On the second assumption, on the contrary—a supposition which is perfectly admissible—*A may change its class.* In the first trial, for example, A may turn out to represent a certainty, in the next a variable, and in the third an impossibility. When a certainty or an impossibility turns up, the statement $C(A)$ is evidently false; when a variable turns up, $C(A)$ is evidently true;

and since (with the data taken) each of these events is possible, and indeed always happens in the long run, $C(A)$ may be false or true, being sometimes the one and sometimes the other, and is therefore a variable. That is to say, on perfectly admissible assumptions, $CC(A)$ is possible; it is not a *formal* impossibility.

But, *with other data*, $C(A)$ may be either a certainty or an impossibility, in either of which cases $CC(A)$ would be an impossibility. For example, if all the statements from which A is taken at random be exclusively variable, ... then, evidently, we should have $NC(A)$, and not $CC(A)$. On the other hand, if our universe of statements consisted solely of certainties and impossibilities, with no variables, we should have $IC(A)$, and not $CC(A)$. Thus the statement $CC(A)$ is *formally* possible; that is to say, it contradicts no definition or linguistic or symbolic convention; but whether or not it is *materially* possible depends upon our special or material data. (MacColl 1910, pp. 197–198; MacColl's italics)

The distinction between formal and material possibility is of little help here. On the other hand, MacColl is almost right about the status of $CC(A)$. By definition, this formula means $MC(A) \wedge M\neg C(A)$. The second conjunct, that is, $M\neg C(A)$ is equivalent to $M(N(A) \vee I(A))$, which gives that $CC(A)$ is equivalent to $MC(A) \wedge M(N(A) \vee I(A))$. Thus, $CC(A)$ says that it is possible that A is contingent and it is possible that A is necessary or impossible. Now it is evident that the formula $CC(A)$ is either true or false, depending on the status of A. If A is a possibility, then it is possible (not excluded) that $C(A)$ and $C(\neg A)$, so $MC(A)$ is true. Since $M(A)$ does not exclude $N(A)$, then if A is possible, it is possible that A is necessary and the second conjunct is also true. On the other hand, if A is either necessary or impossible, the formula $MC(A)$, that is, the first conjunct of $MC(A) \wedge M(N(A) \vee I(A))$ is false, and the whole formula $CC(A)$ is false. It seems that complications introduced by MacColl are caused by his confusing contingency and possibility. This confusion seems to me more important than other unclarities pointed out by Shearman (1906, pp. 152–161), who argued that MacColl did not observe that certainty implies truth, that he confused events and statements as well as propositions and propositional functions, and that he misinterpreted other logicians as far as relations between particular modalities were concerned. I will not discuss Shearman's objections (they were directed at papers preceding MacColl 1906, which clarified some points), because the formal connections between modals that MacColl noted are fortunately independent of a particular interpretation of modalities and the distinction of propositions and propositional functions. Thus, MacColl's work can be rightly regarded as a predecessor of the formal logic of modalities.

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References

Dejnožka, J. 1999. *Bertrand Russell on Modality and Logical Relevance.* Ashgate, Aldershot.

Frege, G. 1879. *Begriffsschrift, eine der Arithmetischen Nachgebildete Formelsprache des Reinen Denkens.* Verlag von Louis Nebert, Halle. English translation in van Heijenoort 1967 as "Begriffsschrift, A formula language, modeled upon that of arithmetic, for pure thought".

van Heijenoort, J. (ed.). 1967. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931.* Harvard University Press, Cambridge, Mass.

Lewis, C. I. 1912. Implication and the algebra of logic. *Mind,* n.s., vol. 21, pp. 522–531.

———. 1918. *A Survey of Symbolic Logic.* University of California Press, Berkeley.

MacColl, H. 1906. *Symbolic Logic and Its Applications.* Longmans, Green, and Co.

———. 1910. Linguistic misunderstandings (I). *Mind,* n.s., vol. 19, pp. 186–199.

Rescher, N. 1974. Bertrand Russell and modal logic. In N. Rescher (ed.), *Studies in Modality*, pp. 85–96. Basil Blackwell, London.

Russell, B. 1903. *The Principles of Mathematics.* Cambridge University Press, Cambridge.

———. 1905. On denoting. *Mind,* n.s., vol. 14, pp. 479–493.

———. 1919. *Introduction to Mathematical Philosophy.* George Allen and Unwin, London.

Shearman, A. T. 1906. *The Development of Symbolic Logic.* Williams and Norgate, London.

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