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HUGH MACCOLL AND THE ALGEBRA OF STRICT IMPLICATION

C. I. Lewis repeatedly exempts MacColl from criticisms of his predecessors in their accounts of implication. They had all taken a true implication, or conditional, to be one with false antecedent or true consequent. MacColl uniquely, and correctly in Lewis' view, rejected this account, identifying a true implication with the impossibility of true antecedent and false consequent. Lewis' development of the calculus of strict implication arises directly and explicitly out of MacColl's work.

A close analysis of MacColl's calculatory methods, and summaries of his main theses, serve to show that MacColl's modal logic is in fact the logic T introduced by Feys and von Wright many decades later, the smallest normal epistemic modal logic.

1. MacColl and Lewis

The received wisdom is that strict implication was invented and developed by the American logician C. I. Lewis. Careful reading of Lewis' papers, and of his book, *A Survey of Symbolic Logic* (1918), reveals that he repeatedly exempts MacColl from criticisms of his predecessors in their accounts of implication. From this fact, it is clear that Lewis knew MacColl's work; he is not fully candid, however, in acknowledging his debt to MacColl. Indeed, in later life Lewis seemed to take great pains to obscure the origins of his modal calculi as they are presented in his joint work with Langford (1932). Thus, for the 1960 Dover reprint of Lewis 1918, a whole third of the book (chapters 5 and 6), containing Lewis' first treatment of modal logic in book form, was completely omitted at Lewis' instigation (the ground being that what was said there had been superseded by the later treatment—and indeed, the system of the *Survey* was flawed, in containing too powerful a form of the Consistency Postulate). Furthermore, in collecting his articles for their 1970 reprinting (Lewis 1970), Lewis omitted all

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but one of the papers published prior to 1918 on the notion of strict implication. The result was a wholesale removal of many of what brief acknowledgements of MacColl there were in Lewis' writings.

MacColl suffered, however, from an even greater eclipsing of his logical contribution than simply from being excluded from Lewis' revisionist history. In the second half of the nineteenth century, the dominant programme in logic was the Boole-Schröder algebraic system, construing the logical constants as operators in a class algebra. The first two volumes of Schröder's famous *Lectures on the Algebra of Logic* (1890–1905) are studded with (complimentary) references to MacColl. As we will see, MacColl recognized that one had to supplement the existing extensional algebras (extensional in that they admitted a class interpretation) with a further (intensional) operator in order to have any prospect of properly capturing the logic of implication. Within months of MacColl's death in December 1909 came the publication of Whitehead and Russell's *Principia Mathematica* (1910–1912). The result was the rapid replacement of the Boole-Schröder algebraic paradigm by the logistic methods developed by Frege, Peano and others. Only a few years after MacColl's death, the method of systematic proof from axioms of a logistic formulation had replaced the algebraic methods of calculation of the nineteenth century. Lewis caught the spirit of the times, recasting MacColl's modal calculus in logistic terms. I do not want to deny that Lewis provided deep logical insights in his presentation of strict implication in the fashionable new guise. But again, the presentational novelty served to obscure MacColl's contribution from all but the keenest observers.

MacColl's logic was developed in several series of papers whose driving methodology is the calculatory and applied paradigm of the late nineteenth century. The preferred argument for a logical method was its success in application to specific problems. MacColl repeatedly claims superiority of his calculus over that of the "Boolian logicians" as he calls them (Jevons, Schröder et al.) on the grounds of its greater success in solving specific problems from the *Educational Times* or the *Proceedings of the London Mathematical Society*.[1] The development of the background theory, and in particular, the systematic specification of that theory in axiomatic and deductive terms, was a requirement that only came later, following the development of the logistic method.

MacColl saw his calculus as differing in two main regards from the "Boolian" calculus. First, he emphasized its propositional interpretation as at least as important as the customary class interpretation.

[1]See, e.g., MacColl 1903b, cited in MacColl 1998, 15 May 1905.

In fact, one of MacColl's repeated themes is the preference for multiplicity of interpretation. MacColl wrote (1902, p. 362):

> Perhaps the most important principle underlying my system of notation is the principle that we may vary the meaning of any symbol or arrangement of symbols, provided, firstly, we accompany the change of signification by a new explanatory definition; and provided, secondly, the nature of our argument be such that we run no risk of confounding the old meaning with the new. Of course this variation of sense should not be resorted to wantonly and without cause; but the cases are numerous in which it leads both to clearness of expression and to an enormous economy in symbolic operations.

Thus his calculus admits of a class interpretation, a propositional interpretation and an interpretation in probability theory. But the propositional interpretation was the novelty for such an algebraic system, and the one MacColl emphasized as distinctive of his approach.

Secondly, he claimed that the two alethic modalities, 'true' and 'false', symbolized by '1' and '0' in the "Boolian" system, were inadequate to deal with all problems arising in mathematics. Three further modalities he introduced were 'certain' (or 'necessary'), 'impossible' and 'contingent'. One motivation for this was the probabilistic interpretation. A true proposition can have any probability value greater than 0, not necessarily 1. Again, a false proposition can have any value less than 1, not necessarily 0. When we discover that the probability of a proposition is 1, we know it is certain, not just that it is true; when we find its probability is 0, we know it is impossible, not just false. Of course, these values are relative to certain evidence, so we know that the proposition is certain, impossible or neither, relative to certain data. MacColl first introduced ϵ (for certainty), η (for impossibility) and θ (for variability, or contingency) relative to the data. Subsequently, he generalized them to stand also for certainty *tout court*, that is, necessity, for impossibility and for contingency. Thus a^ϵ reads 'a is certain (or necessary)', a^η reads 'a is impossible' and a^θ reads 'a is variable (or contingent)'.[2]

My aim in this paper is to give a systematic presentation of MacColl's modal algebra. It is based on Boole's algebra, extending it by the alethic modalities and strict implication as further operators. Although, as we will see, only one of these need be taken as primitive, the rest being definable, I will take both possibility and strict implication as primitive. The situation is not unlike that in Boolean algebras, which take meet, join and complement as primitive, though meet and join can each be defined in terms of the other and complement.

[2]See, e.g., MacColl 1901, § 3, p. 138. He also wrote a^τ for 'a is true' and a^ι for 'a is false'.

I will start with a reminder of the formal theory of Boolean algebras, presented in a more systematic way than was common in MacColl's time. This presentation will also serve to exhibit the terminology and methodology of proof and demonstration.

2. Boolean Algebras

An algebra consists of a set of elements closed under one or more operations satisfying certain conditions. The idea of algebraic logic is to define a class of algebras which characterize logical validity (or more generally, logical consequence). For example, Boolean algebras characterize classical (propositional) logic, pseudo-Boolean (sometimes called Heyting) algebras characterize intuitionistic logic, and cylindric algebras characterize full (first-order) classical logic. In the Russell/Whitehead paradigm, a logic is taken to be a class of wffs and a subset of validities (or better, a consequence relation on those wffs). The well-formed formulae of the logic are mapped to elements of the algebra via a homomorphism[3] defined on the atomic wffs which identifies equivalent wffs. Much of algebraic logic then consists of the identification of the algebra of equivalent wffs (the Lindenbaum algebra), showing that it is free[4] in a certain class of algebras, that the set of validities is a filter[5] (commonly, as in the cases cited above, the trivial filter consisting of the maximum element alone), and studying homomorphisms from the Lindenbaum algebra to more manageable, in particular, finite algebras in the class.

In the Boole-Schröder paradigm, the representation of situations goes directly to the elements of the algebra, omitting the syntactic intermediary of a language of wffs. The emphasis is on exploration of the algebraic structure and solution of mathematical (and other) problems, rather than the metalogical analysis and representation theory more common today. Logical algebra was seen as a further mathematical tool which proved itself by its utility. The algebras were studied as individuals of a type whose properties were developed piecemeal, rather than systematically as a class. The whole approach was—at least when compared with that which replaced it—calculatory and unsystematic.

[3]A homomorphism is a mapping between algebras of the same type which preserves the operations, e.g., if $\circ$ is a two-place operation on A matching a similar operation on B, and h is a homomorphism from A to B, then $h(a \circ b) = ha \circ hb$.

[4]An informal account of when an algebra is free in a class is if it satisfies no further conditions than those on the class as a whole.

[5]A filter F in a lattice (see below) is a subset such that $a \cap b \in F$ iff $a \in F$ and $b \in F$.

The insight which the two paradigms share, however, is recognition that logical structure responds productively to the application of algebraic techniques.

A Boolean algebra is a complemented distributive lattice. Some authors take lattices to be a particular type of poset (partially ordered set), one in which every two-element subset has a *sup* (supremum) and *inf* (infimum). However, I want to present MacColl's ideas purely algebraically, so I will take lattices to be algebras, even though, as we will see, they can also be viewed as relational structures of a certain sort. I will by and large follow MacColl's notation, in particular, taking . (or concatenation) and + for meet and join respectively, and $'$ for complement, and writing ϵ and η for the maximum and minimum elements of the algebra.

DEFINITION 2.1. *A* lattice L *consists of a set of elements closed under the operations of meet (.) and join (+): $\langle A, ., + \rangle$, subject to the following constraints:*

$$\begin{aligned} a.(b.c) &= (a.b).c \quad \text{for all } a, b, c \in A. && \text{(Associativity for meet)} \\ a + (b + c) &= (a + b) + c \quad \text{for all } a, b, c \in A. && \text{(Associativity for join)} \\ a.b = b.a,\ a + b &= b + a \quad \text{for all } a, b \in A. && \text{(Commutativity)} \\ a.(a + b) &= a + a.b = a \quad \text{for all } a, b \in A. && \text{(Absorption)} \end{aligned}$$

L *is* non-trivial *if there are $a, b \in L$ such that $a \neq b$.*

LEMMA 2.1. *In any lattice, $a.a = a = a + a$.*[6]

Proof. $a.a = a.(a + a.a) = a = a + a.(a + a) = a + a$. □

DEFINITION 2.2. *A* preordering *on a set X is a relation $\leq$ such that*

$$\begin{aligned} a &\leq a \quad \text{for all } a \in X. && \text{(Reflexivity)} \\ \text{if } a \leq b \text{ and } b \leq c \text{ then } a &\leq c \quad \text{for all } a, b, c \in X. && \text{(Transitivity)} \end{aligned}$$

DEFINITION 2.3. *X is* partially ordered *by $\leq$ if $\leq$ is a preordering on X such that*

$$\text{if } a \leq b \text{ and } b \leq a \text{ then } a = b \quad \text{for all } a, b \in X \qquad \text{(Antisymmetry)}$$

DEFINITION 2.4. *A* poset *is a set X with a partial ordering $\leq$: $\langle X, \leq \rangle$.*

DEFINITION 2.5. *1. a is an* upper bound *of $A \subseteq X$ if for all $b \in A$, $b \leq a$.*

2. a is a lower bound *of $A \subseteq X$ if for all $b \in A$, $a \leq b$.*

[6] MacColl 1896, p. 178 (21).

Definition 2.6. *1. a is the* supremum *(or* least upper bound*) of $A \subseteq X$ if a is an upper bound of A and for all $b \in X$, if b is an upper bound of A then $a \leq b$. We write $a = sup(A)$.*

2. a is the infimum *(or* greatest lower bound*) of $A \subseteq X$ if a is a lower bound of A and for all $b \in X$, if b is a lower bound of A then $b \leq a$. We write $a = inf(A)$.*

Lemma 2.2. *$a.b = a$ iff $a + b = b$.*

Proof. Suppose $a.b = a$. Then

$$\begin{aligned} b &= b + (a.b) && \text{Absorption} \\ &= b + a \\ &= a + b && \text{Commutativity} \end{aligned}$$

Converse: similar. □

Definition 2.7. *Given a lattice L, define $\leq$ on L by: $a \leq b$ iff $a.b = a$.*

Lemma 2.3. *In any lattice, if $a \leq b$ then $a.c \leq b.c$ and $a+c \leq b+c$.*

Proof. Suppose $a \leq b$. Then $a.b = a$ and by Lemma 2.2, $a + b = b$. So $(a.c).(b.c) = (a.b).(c.c) = a.c$. So $a.c \leq b.c$.

Similarly, $(a + c) + (b + c) = (a + b) + (c + c) = b + c$. So again by Lemma 2.2, $a + c \leq b + c$. □

Theorem 2.1. *Each lattice $\langle L, ., + \rangle$ induces a poset $\langle L, \leq \rangle$ in which every pair of elements has a sup and an inf, and vice versa.*

Proof. First, we show that $\leq$ is a p.o. on L.

1. $\leq$ is reflexive, by Lemma 2.1.
2. Suppose $a \leq b$ and $b \leq c$, i.e., $a.b = a$ and $b.c = b$.
   Then $a.c = (a.b).c = a.(b.c) = a.b = a$
   i.e., $a \leq c$. So $\leq$ is transitive.
3. Suppose $a \leq b$ and $b \leq a$, i.e., $a.b = a$ and $b.a = b$.
   Then $$\begin{aligned} a = a.b &= b.a && \text{Commutativity} \\ &= b. && \text{So } \leq \text{ is anti-symmetric.} \end{aligned}$$

Next we show that $a + b = sup\{a, b\}$. Note that $a = a.(a + b)$, so $a \leq a + b$; and that

$$\begin{aligned} b &= b.(b + a) \\ &= b.(a + b), \text{ so } b \leq a + b. \end{aligned}$$

Now, suppose $a \leq c$ and $b \leq c$. Then $a.c = a$ and $b.c = b$. So $a + c = c$ and $b + c = c$, by Lemma 2.2, whence $(a + b) + c = a + (b + c) = a + c = c$, and so $(a + b).c = a + b$, by Lemma 2.2, i.e., $a + b \leq c$. So $a + b = sup\{a, b\}$.

Similarly, $a.b = inf\{a, b\}$.

Conversely, given a poset P in which every pair of elements has a *sup* and an *inf*, define $a.b$ as $inf\{a, b\}$ and $a + b$ as $sup\{a, b\}$. Clearly, on this definition, meet and join are associative and commutative. For absorption, since $a \leq sup\{a, b\}$, $a = inf\{a, sup\{a, b\}\} = a.(a + b)$, and similarly, $a = sup\{a, inf\{a, b\}\} = a + (a.b)$. □

LEMMA 2.4. *Take any lattice, L. If $z \leq x$ iff $z \leq y$ for all $z \in L$, then $x = y$.*

Proof. Since $x \leq x$, $x \leq y$ and since $y \leq y$, $y \leq x$. So as in Theorem 2.1, $x = y$. □

LEMMA 2.5. *In any lattice,*

$$(a.b) + (a.c) \leq a.(b + c)$$

and

$$a + (b.c) \leq (a + b).(a + c).$$

Proof. $a.b \leq a.(b + c)$ and $a.c \leq a.(b + c)$, Lemma 2.3
so $(a.b) + (a.c) \leq a.(b + c)$. by Theorem 2.1
Moreover, $a \leq a + b$ and $a \leq a + c$, by Theorem 2.1
so $a \leq (a + b).(a + c)$. Theorem 2.1
Furthermore, $b \leq a + b$ and $c \leq a + c$,
so $b.c \leq (a + b).(a + c)$, by Lemma 2.3
so $a + (b.c) \leq (a + b).(a + c)$. by Theorem 2.1. □

DEFINITION 2.8. *A lattice is* distributive *if*

$$a.(b + c) = (a.b) + (a.c).^{7} \qquad \textit{(Distributive law of meet over join)}$$

LEMMA 2.6. *In a distributive lattice,*

$$a + (b.c) = (a + b).(a + c). \qquad \textit{(Distributive law of join over meet)}$$

Proof. $(a + b).(a + c) = (a + b).a + (a + b).c = a + c.(a + b)$
$= a + (c.a) + c.b = a + b.c.$ by Lemma 2.3
So by Lemma 2.5, $a + b.c = (a + b).(a + c)$. □

In fact, either distributive law may be derived from the other.

DEFINITION 2.9. *A lattice L has a* maximum *(ϵ) if for all $a \in L$, $a \leq \epsilon$, and it has a* minimum *(η) if for all $a \in L$, $\eta \leq a$.*

DEFINITION 2.10. *A lattice L is* complemented *if for all $a \in L$ there is $b \in L$ such that $a + b = \epsilon$ and $a.b = \eta$.*

LEMMA 2.7. *In a distributive lattice, complements are unique.*

Proof. Suppose a has complements b and c, i.e.,

$$a + b = \epsilon = a + c$$

and

[7] MacColl 1901, p. 141.

$$a.b = \eta = a.c.$$

Then $b = b.(b+a) = b.(a+b) = b.(a+c) = ba+bc = ab+bc = ac+bc = (a+b).c = (a+c).c = c.(c+a) = c.$ □

THEOREM 2.2.

1. $a.\epsilon = a$;[8]
2. $a.\eta = \eta$;[9]
3. $\epsilon + a = \epsilon$;[10]
4. $\eta + a = a$.[11]

Proof.

1. Since $a \leq \epsilon$, $a.\epsilon = a$.
2. Since $\eta \leq a$, $a.\eta = \eta$.
3. Since $a \leq \epsilon$, $\epsilon + a = \epsilon$ by Lemma 2.2.
4. Since $\eta \leq a$, $\eta + a = a$ by Lemma 2.2. □

DEFINITION 2.11. *A* Boolean algebra *consists of a set of elements closed under meet, join and complement (′): $\langle A, ., +, ', \eta, \epsilon\rangle$, such that $\langle A, ., +\rangle$ is a distributive lattice with maximum and minimum, and*

$$a.a' = \eta \textit{ and } a + a' = \epsilon \textit{ for all } a \in A.$$

THEOREM 2.3. *In a Boolean algebra,*

1. $a.b' = \eta$ *iff* $a \leq b$
2. $a + b' = \epsilon$ *iff* $b \leq a$
3. $a'' = a$.

Proof.

1. Suppose $a.b' = \eta$. Then
$$\begin{aligned} a &= a.\epsilon \\ &= a.(b+b') \\ &= (a.b) + (a.b') \\ &= a.b. \text{ So } a \leq b. \end{aligned}$$
Conversely, suppose $a \leq b$. Then $a.b' \leq b.b'$ by Lemma 2.3 $= \eta$. So $a.b' = \eta$.

[8] MacColl 1906, p. 8 (22).
[9] MacColl 1906, p. 8 (23).
[10] MacColl 1901, p. 143 (5).
[11] MacColl 1901, p. 143 (6).

2. The dual case is similar.[12]
3. Immediate from Lemma 2.7. □

THEOREM 2.4.

1. $(a.b)' = a' + b'$;[13]
2. $(a + b)' = a'.b'$;[14]
3. $a \leq b$ *iff* $b' \leq a'$;
4. $(a + b'c)' = a'b + a'c'$.[15]

Proof.

1. $ab(a' + b') = aba' + abb' = \eta + \eta = \eta$ and
   $ab + (a' + b') = (a + a' + b')(b + a' + b') = \epsilon.\epsilon = \epsilon$.
   So $(ab)' = a' + b'$.
2. $a'b'(a + b) = a'b'a + a'b'b = \eta$ and
   $a'b' + (a + b) = (a' + a + b)(b' + a + b) = \epsilon.\epsilon = \epsilon$.
   So $a'b' = (a + b)'$.
3. $a \leq b$ iff $a = a.b$ iff $a' = (a.b)' = a' + b'$ iff $b' \leq a'$.
4. $$\begin{aligned} (a + b'c)' &= a'(b'c)' && \text{by (2)} \\ &= a'(b + c') && \text{by (1)} \\ &= a'b + a'c' && \text{by Definition 2.8.} \end{aligned}$$ □

3. MODAL ALGEBRAS

MacColl's several attempts at systematic presentation of his logic[16] do not satisfy modern standards of rigour. His various statements make clear what theses his algebra contains; what is harder to ascertain is what it does not contain, that is, precisely how strong it is. Hughes and Cresswell (1996) repeat the question they raised in their original text (1968):

> MacColl does give a list of 'self-evident formulae' and it would be interesting to know which of the more recent modal systems is the weakest in which all these are true. (Hughes and Cresswell 1968, p. 214 n. 177; 1996, p. 206 n. 4)

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[12] Let P be any statement about lattices, Boolean algebras, etc. If in P we replace $\leq$ by $\geq$, . by +, + by ., and each element a by a' (and a' by a), we obtain the dual (statement) P'. Then P is true iff P' is true.

[13] MacColl 1901, p. 141; MacColl 1906, p. 8 (2).

[14] MacColl 1901, p. 141; MacColl 1906, p. 8 (3).

[15] MacColl 1906, p. 9 (1).

[16] E.g., MacColl 1901, 1906.

My claim is that MacColl's modal algebra is what has later come to be called the normal modal logic T. The algebraic treatment of T along with other weak modal logics was presented in Lemmon 1966, building on work in Lemmon 1960, developing, for the modal logics T, S2, S3 and so on, what he called "extension algebras" generalizing the closure algebras of McKinsey and Tarski 1944. The system T was characterized as that of normal epistemic extension algebras. I will call extension algebras (including closure algebras), modal algebras.

Definition 3.1. *A* modal algebra *consists of a set of elements closed under meet, join, complement and extension (possibility,* $^\pi$*):* $\langle A, ., +, ', ^\pi, \eta, \epsilon \rangle$ *such that*

1. $\langle A, ., +, ', \eta, \epsilon \rangle$ *is a Boolean algebra, and*
2. $(a + b)^\pi = a^\pi + b^\pi$. *(K)*

Definition 3.2. *A modal algebra is* epistemic *if it also satisfies the postulate*

$$a \leq a^\pi. \qquad (T)$$

Definition 3.3. *A modal algebra is* normal *if it also satisfies the postulate*

$$\eta^\pi = \eta. \qquad (N)$$

Definition 3.4. *Let*

$$a^\eta = a^{\pi\prime}, \qquad a^\epsilon = a^{\prime\eta} \qquad \textit{and} \qquad a^\theta = (a^\epsilon + a^\eta)'.$$

Lemma 3.1.

1. $a^\pi = a^{\prime\epsilon\prime}$;
2. $a^\epsilon = a^{\prime\pi\prime}$.

Proof.

1. $$\begin{aligned} a^{\prime\epsilon\prime} &= a^{\prime\prime\eta\prime} && \text{by Definition 3.4 (2)} \\ &= a^{\eta\prime} && \text{since } a'' = a \\ &= a^{\pi\prime\prime} && \text{by Definition 3.4 (1)} \\ &= a^\pi. \end{aligned}$$
2. $$\begin{aligned} a^{\prime\pi\prime} &= a^{\prime\eta} && \text{by Definition 3.4 (1)} \\ &= a^\epsilon && \text{by Definition 3.4 (2).} \end{aligned}$$ □

THEOREM 3.1.

1. $(a.b)^\epsilon = a^\epsilon.b^\epsilon$;[17]
2. $(a+b)^\theta = (a'b')^\theta$;[18]
3. *if* $a \leq b$ *then* $a^\pi \leq b^\pi$ *and* $a^\epsilon \leq b^\epsilon$;[19]
4. $a^\epsilon \leq a$.[20]

Proof.

1. $$\begin{aligned}(a.b)^\epsilon = (a.b)'^\eta = (a'+b')^\eta &= (a'+b')^{\pi\prime} \\ &= (a'^\pi + b'^\pi)' \qquad \text{by } (K) \\ &= a'^{\pi\prime}.b'^{\pi\prime} = a^\epsilon.b^\epsilon.\end{aligned}$$

2. $(a+b)^\theta = ((a+b)^\epsilon + (a+b)^\eta)' = (a+b)^{\epsilon\prime}.(a+b)^{\eta\prime} = (a+b)'^\pi.(a+b)^\pi = (a'b')^\pi.(a'b')'^\pi = (a'b')'^\pi.(a'b')^\pi = (a'b')^{\epsilon\prime}.(a'b')^{\eta\prime} = ((a'b')^\epsilon + (a'b')^\eta)' = (a'b')^\theta$.

3. Suppose $a \leq b$. Then $a+b=b$ and $a.b=a$.
Hence $b^\pi = (a+b)^\pi = a^\pi + b^\pi$ by (K)
i.e., $a^\pi \leq b^\pi$
and $a^\epsilon = (a.b)^\epsilon = a^\epsilon.b^\epsilon$ by (1)
i.e., $a^\epsilon \leq b^\epsilon$.

4. $a' \leq a'^\pi$ by (T)
so $a'^{\pi\prime} \leq a''$ by Theorem 2.4 (3)
i.e., $a^\epsilon \leq a$. □

THEOREM 3.2. [21]

1. $\epsilon^\eta = \eta$;
2. $\eta^\epsilon = \eta$;
3. $\eta^\eta = \epsilon$;[22]
4. $\epsilon^\epsilon = \epsilon$;
5. $\epsilon^\theta = \eta$.

[17] MacColl 1896, p. 169 ; cf. MacColl 1906, p. 72 (7).
[18] MacColl 1896, p. 169 .
[19] MacColl 1906, p. 9 § 13.
[20] Cf. MacColl 1906, p. 8 (15), which reads $A^\epsilon : A^\tau$, meaning 'If A is certain, then A is true'—see §4 below. MacColl writes (op.cit. § 8, p. 7) that A^ϵ asserts more than A^τ, which "only asserts that A is true in a particular case or instance." A^ϵ asserts "that A is *certain*, that A is *always* true (or true in *every* case)."
[21] MacColl 1901, p. 140.
[22] See also MacColl 1998, 6 Oct 1901.

Proof.

1. By (T), $\epsilon \leq \epsilon^\pi$. So $\epsilon^\eta = \epsilon^{\pi\prime} \leq \epsilon' = \eta$. Hence $\epsilon^\eta = \eta$.
2. $\eta^\epsilon = \eta'^\eta = \epsilon^\eta = \eta$ by (1).
3. $\eta^\eta = \eta^{\pi\prime} = \eta'$ (by N) $= \epsilon$.
4. $\epsilon^\epsilon = \epsilon'^\eta = \eta^\eta = \epsilon$ by (3).
5. $\epsilon^\theta = (\epsilon^\epsilon + \epsilon^\eta)' = (\epsilon + \eta)'$ by (1) and (4) $= \epsilon' = \eta$. □

Theorem 3.3.

1. $(a + a')^\epsilon = \epsilon$;[23]
2. $(a.a')^\eta = \epsilon$;[24]
3. $(a^\epsilon + a^\eta + a^\theta)^\epsilon = \epsilon$.[25]

Proof.

1. $(a + a')^\epsilon = \epsilon^\epsilon = \epsilon$.
2. $(a.a')^\eta = \eta^\eta = \epsilon$.
3. $(a^\epsilon + a^\eta + a^\theta)^\epsilon = ((a^\epsilon + a^\eta) + (a^\epsilon + a^\eta)')^\epsilon = \epsilon^\epsilon = \epsilon$. □

Theorem 3.4. *Where* $\circ$ *is* $^\pi$, $^\epsilon$, $^\eta$, $^\theta$, *let* $a^{-\circ} = a^{\circ\prime}$. *Then*[26]

1. $a^\theta a^{-\theta} = \eta$;
2. $a^{-\theta} = a^\epsilon + a^\eta$;
3. $a^{-\epsilon} = a^\eta + a^\theta$;
4. $(a^\epsilon + b^{-\epsilon}c^\epsilon)' = (a^\eta + a^\theta)(b^\epsilon + c^\eta + c^\theta)$;
5. $(a^{-\theta} + a^\theta b^\theta)' = a^\theta(b^\epsilon + b^\eta)$.

Proof.

1. $a^\theta a^{-\theta} = a^\theta a^{\theta\prime} = \eta$.
2. $a^{-\theta} = a^{\theta\prime} = (a^\epsilon + a^\eta)'' = a^\epsilon + a^\eta$.
3. By Theorem 3.1 (4), $a^\epsilon \leq a$
   and by T, $a \leq a^\pi$, so $a^\epsilon \leq a^\pi$ by the proof of Theorem 2.1,
   i.e., $a^\epsilon = a^\pi.a^\epsilon = a^\pi.a^\epsilon + \eta = a^\pi.a^\epsilon + a^\pi.a^\eta = a^\pi(a^\epsilon + a^\eta) =$
   $(a^\eta + (a^\epsilon + a^\eta)')' = (a^\eta + a^\theta)'$.
   So $a^{-\epsilon} = a^\eta + a^\theta$.

[23] MacColl 1896, p. 177 (2).
[24] MacColl 1906, p. 8 (13).
[25] MacColl 1906, p. 8 (14).
[26] MacColl 1906, p. 9.

4. $(a^\epsilon + b^{-\epsilon}c^\epsilon)' = a^{\epsilon\prime}.(b^{-\epsilon}c^\epsilon)' = a^{-\epsilon}.(b^{\epsilon\prime\prime} + c^{\epsilon\prime})$
$$= (a^\eta + a^\theta).(b^\epsilon + c^{-\epsilon}) \quad \text{by (3)}$$
$$= (a^\eta + a^\theta).(b^\epsilon + c^\eta + c^\theta). \quad \text{by (3) again.}$$
5. $(a^{-\theta} + a^\theta b^\theta)' = a^{\theta\prime\prime}.(a^\theta b^\theta)' = a^\theta.(a^{-\theta} + b^{-\theta}) = a^\theta.a^{-\theta} + a^\theta.b^{-\theta} = \eta + a^\theta.b^{-\theta} = a^\theta(b^\epsilon + b^\eta)$. □

4. MacColl Algebras

A MacColl algebra is, in essence, a normal epistemic modal algebra (or a T-algebra, for short). However, as we have noted, MacColl adds a further operator, a conditional operator, to his algebras. Thus we can best represent his algebra as a T-algebra with a further conditional operator, ':'.

Definition 4.1. *A* MacColl algebra *consists of a normal epistemic modal algebra equipped with a conditional operator, :, i.e., a structure* $\langle A, ., +, ', ^\pi, :, \eta, \epsilon\rangle$ *such that*

$$a : b = (ab')^\eta \qquad (SI)$$

$a : b$ represents strict implication.

Theorem 4.1.

1. $a : b = b' : a'$;[27]
2. $a : b = (a' + b)^\epsilon$;
3. $(x : a)(x : b) = x : ab$;[28]
4. $(a + b) : x = (a : x)(b : x)$.[29]

Proof.

1. $a : b = (ab')^\eta = (b'a'')^\eta = b' : a'$.
2. $a : b = (ab')^\eta = (ab')^{\prime\epsilon} = (a' + b)^\epsilon$.
3. $(x : a)(x : b) = (x' + a)^\epsilon.(x' + b)^\epsilon$
$$= [(x' + a).(x' + b)]^\epsilon \quad \text{by } (K)$$
$$= (x' + ab)^\epsilon \quad \text{by Lemma 2.6}$$
$$= x : ab.$$
4. $(a : x)(b : x) = (a' + x)^\epsilon.(b' + x)^\epsilon = [(a' + x).(b' + x)]^\epsilon = (a'b' + x)^\epsilon = ab : x$. □

[27] MacColl 1901, p. 144 (7); MacColl 1906, p. 8 (4).
[28] MacColl 1906, p. 8 (5).
[29] MacColl 1906, p. 8 (6).

LEMMA 4.1.

1. *$a : b = \epsilon$ iff $a \leq b$.*
2. *Let $a :: b =_{df} (a : b)(b : a)$. Then $a :: b = \epsilon$ iff $a = b$.*

Proof.

1. $a \leq b$ iff $ab' = \eta$ by Theorem 2.3. Suppose $ab' = \eta$. Then $a : b = (ab')^\eta = \eta^\eta = \epsilon$.
   Conversely, suppose $a : b = \epsilon$. Then $(ab')^\eta = \epsilon$, so $(ab')^\pi = \eta$. But $ab' \leq (ab')^\pi$ by T, so $ab' = \eta$.
2. if $a :: b = \epsilon$ then $a : b = b : a = \epsilon$, so $a \leq b$ and $b \leq a$, whence $a = a.b = b$.
   Conversely, if $a = b$ then $a :: b = a :: a = a : a = (aa')^\eta = \eta^\eta = \epsilon$. □

THEOREM 4.2.

1. *$a^\epsilon = a :: \epsilon$;*[30]
2. *$a : \epsilon = \epsilon$;*[31]
3. *$a^\eta = a :: \eta = a : \eta$.*[32]

Proof.

1. $a :: \epsilon = (a : \epsilon)(\epsilon : a) = (a\epsilon')^\eta(\epsilon a')^\eta = (a\eta)^\eta(\epsilon a')^\eta$
   $= \eta^\eta(\epsilon a')^\eta = \epsilon(\epsilon a')^\eta = (\epsilon a')^\eta = a'^\eta = a^\epsilon$.
2. $a : \epsilon = (a\epsilon')^\eta = (a\eta)^\eta = \eta^\eta = \epsilon$.
3. $a :: \eta = (a : \eta)(\eta : a) = (a\eta')^\eta(\eta a')^\eta = a^\eta.\eta^\eta = a^\eta = (a\eta')^\eta$
   $= a : \eta$. □

LEMMA 4.2. *Let $a \supset b =_{df} a' + b$ ($\supset$ is material implication). Then*

$$a.b \leq c \text{ iff } a \leq b \supset c.$$

Proof. Suppose $a.b \leq c$.

$$\begin{aligned} \text{Then } a = a(b' + b) = ab' + ab &= a(b' + ab) \\ &\leq b' + a.b \leq b' + c \quad \text{by Lemma 2.3} \\ &= b \supset c. \end{aligned}$$

Conversely, suppose $a \leq b \supset c = b' + c$.

$$\begin{aligned} \text{Then } a.b &\leq (b' + c).b \quad \text{by Lemma 2.3} \\ &= b'.b + c.b = b.c \leq c. \end{aligned}$$ □

[30] MacColl 1901, p. 144 (10).
[31] MacColl 1998, 6 Oct 1901.
[32] MacColl 1901, p. 144 (11).

LEMMA 4.3. $(a \supset b)^\epsilon \leq a^\epsilon \supset b^\epsilon$.

Proof. Note that $a.(a' + b) = a.a' + a.b = ab \leq b$.

So $a^\epsilon.(a'+b)^\epsilon = (a.(a'+b))^\epsilon$ by Theorem 3.1 (1)

$\leq b^\epsilon$ by Theorem 3.1 (3).

So $(a \supset b)^\epsilon = (a'+b)^\epsilon \leq a^\epsilon \supset b^\epsilon$ by Lemma 4.2. □

THEOREM 4.3. $a^\epsilon.(a : b) \leq b^\epsilon$.[33]

Proof. By Lemma 4.3, $a : b \leq a^\epsilon \supset b^\epsilon$.

So by Lemma 4.2, $a^\epsilon.(a : b) \leq b^\epsilon$. □

We have now shown that MacColl's logic was at least as strong as the modal logic T. The three principles which are crucial to this are:

Theorem 3.1 (1)	$(a.b)^\epsilon = a^\epsilon.b^\epsilon$	i.e., K
Theorem 3.1 (4)	$a^\epsilon \leq a$	i.e., T
Theorem 3.2 (3)	$\eta^\eta = \epsilon$	i.e., N.

Let us show that these results are each equivalent to the principles stated. We can see from the results adduced that each of the principles K, T and N entails the results given. Conversely: first, suppose

$$(a.b)^\epsilon = a^\epsilon.b^\epsilon. \tag{*}$$

Then $(a+b)^\pi = (a'b')'^\pi = (a'b')^{\epsilon\prime} = (a'^\epsilon.b'^\epsilon)'$ by (*)

$= (a^\eta b^\eta)' = a^{\eta\prime} + b^{\eta\prime} = a^\pi + b^\pi$,

i.e., (*) entails K.

Next, suppose

$$a^\epsilon \leq a. \tag{**}$$

We need to derive T, *viz* $a \leq a^\pi$. Substituting a' for a in (**), we have $a'^\epsilon \leq a'$, so by Theorem 2.4 (3), $a'' \leq a'^{\epsilon\prime}$, whence $a \leq a^\pi$ by Theorem 2.3 (3) and Theorem 3.1 (1).

Finally, suppose

$$\eta^\eta = \epsilon. \tag{***}$$

We need to derive N, *viz* $\eta^\pi = \eta$.

From (***), $\eta = \epsilon' = \eta^{\eta\prime} = \eta^{\pi\prime\prime}$ by Definition 3.4 (1)

$= \eta^\pi$.

Might MacColl's calculus be stronger than T? There is good reason to think not. For T is among the strongest systems in which there are infinitely many modalities. Any stronger system would contain reduction laws, such as $a^{\pi\pi} = a^\pi$. But MacColl makes no reference

[33] MacColl 1906, p. 9 § 13; cf. Spencer 1973, p. 57 (10).

to any such reduction.[34] Note that η and ϵ behave differently when used as exponents and as formulae themselves. For $a^{\eta\epsilon}$, for example, means $(a^\eta)^\epsilon$, not $a^{(\eta\epsilon)}$, so the fact that, say, $\eta\epsilon = \eta$ is irrelevant to such possible reductions of exponents. In MacColl 1903a, p. 361, he considers the formula $a^{\theta\theta\epsilon} + a^{\theta\theta\eta} + a^{\theta\theta\theta}$, but makes no suggestion that the complex modalities can be reduced. In fact, in MacColl 1897 he explicitly rejects $a^\epsilon : a^{\epsilon\epsilon}$ (the characteristic axiom of S4) and its like:

> when the statement α or β may belong sometimes to one and sometimes to another of the three classes ϵ, η, θ, the formulae $(\alpha : \beta)^\epsilon : (\alpha : \beta)$ and $(\alpha : \beta)^\eta : (\alpha : \beta)'$ will of course still be valid, but *not* always the converse formulae $(\alpha : \beta) : (\alpha : \beta)^\epsilon$ and $(\alpha : \beta)' : (\alpha : \beta)^\eta$. Similarly, we may still accept $\alpha^{\epsilon\epsilon} : \alpha^\epsilon$, $\alpha^{\epsilon\eta} : \alpha^{\epsilon\iota}$ [i.e., $\alpha^{\epsilon\eta} : \alpha^{\epsilon\prime}$], $\alpha^{\eta\eta} : \alpha^{\eta\iota}$, &c., as valid, but *not* their converses $\alpha^\epsilon : \alpha^{\epsilon\epsilon}$, $\alpha^{\epsilon\iota} : \alpha^{\epsilon\eta}$, $\alpha^{\eta\iota} : \alpha^{\eta\eta}$, &c. (MacColl 1897, p. 579)

In none of his calculations does he try to reduce the number of modalities by such laws.

McCall (1967) claims that MacColl's system was "in many respects identical to Lewis' system *S3*" (p. 546). But the characteristic axiom of S3 does not figure in the nine theses McCall attributes to MacColl[35]—indeed, if it did, then since MacColl's logic is normal, as shown above (i.e., $\eta^\pi = \eta$), there would ensue reduction theses such as $a^{\pi\pi} = a^\pi$, characteristic of S4, since S4 is the union of S3 and T. Since MacColl explicitly endorses normality and denies any reduction laws, his logic is T.

5. The Paradoxes of Implication

MacColl introduced his connective ':' out of dissatisfaction with the material implication $\supset$ of the "Boolian" logicians. So it is important to him that his algebraic analysis reject the following formulae:[36]

(1) $$(a : b) + (b : a)$$

and

(2) $$(ab : c) : ((a : c) + (b : c)).$$

We can show, with a suitable MacColl algebra, and suitable assignments to a, b and c, that we can set (1) and (2) different from ϵ.

[34] See Spencer 1973, pp. 26–27. Spencer infers that *if* MacColl's system is any of Lewis', it will be S1–S3. But as we have seen, it is not.

[35] The nine theses either follow immediately from Definition 3.4 or are proved in or follow from Theorems 3.1, 3.3, 4.1 and 4.2.

[36] Spencer 1973, p. 57 (17) and (18).

Let M be based on the Boolean algebra:

$$\begin{array}{ccc} & \epsilon & \\ x & & x' \\ & \eta & \end{array}$$

with operations $^\pi$ and : defined by the tables:

$a : b$	ϵ	x	x'	η	a^π
ϵ	ϵ	x	η	η	ϵ
x	ϵ	ϵ	η	η	ϵ
x'	ϵ	x	ϵ	x	x'
η	ϵ	ϵ	ϵ	ϵ	η

(The table for : is of course derivative from that for $^\pi$.) So $x'^\epsilon = \eta$. M is a MacColl algebra. Let $a = x$ and $b = x'$. Then (1) $(a : b) + (b : a) = (x : x') + (x' : x) = x'^\epsilon + x^\epsilon = \eta + x = x \neq \epsilon$. So (1) is invalid in M.

M also serves to invalidate (2). Let $a = x$, $b = x'$ and $c = \eta$. Then $(ab : c) : ((a : c) + (b : c)) = (xx' : \eta) : ((x : \eta) + (x' : \eta)) = (xx'\eta')^\eta : ((x' + \eta)^\epsilon + (x'' + \eta)^\epsilon) = (\eta\epsilon)^\eta : (x'^\epsilon + x^\epsilon) = \eta^\eta : (\eta + x) = \epsilon : x = (\eta + x)^\epsilon = x^\epsilon = x \neq \epsilon$.[37]

Unsurprisingly, therefore, MacColl's theory of implication avoids the so-called paradoxes of material implication. The following are invalid in M:

(3) $$b : (a : b)$$

and

(4) $$a' : (a : b).$$

In the case of (3), let $a = x$ and $b = x'$ in M; then $b : (a : b) = x' : (x : x') = x' : (x' + x')^\epsilon = x' : x'^\epsilon = x' : \eta = (x'' + \eta)^\epsilon = x^\epsilon = x \neq \epsilon$. The same assignment invalidates (4) as well, for then $a' : (a : b) = x' : (x : x') = x \neq \epsilon$.

MacColl gives natural language examples to support this rejection of material implication as the correct account of implication. He suggests letting a = 'He will persist in his extravagance' and b = 'He will be ruined'. Then (3) is rejected because even if he is ruined, we may

[37] MacColl (1906, § 70 pp. 74–5) gives a counterexample to (2). See also MacColl 1903a, p. 362. Cf. Shearman 1906, pp. 29–30.

still hold that he might have persisted in his extravagance and not have been ruined; and (4) is rejected because, even if he does not persist in his extravagance, we may again hold that he might have persisted and still not have been ruined. On a material account of implication these thoughts are simply contradictory, for $(a \supset b)' = ab'$, which contradicts both b and a', leaving no room to distinguish between supposing he might persist and not be ruined and supposing he does persist and is not ruined. The conditional, says MacColl, contains a modal element, revealed by negating it. $(a : b)' = (ab')^{\eta\prime} = (ab')^{\pi}$, that is, it is possible that a (he persists) and b' (he is not ruined).

Nonetheless, this analysis does open MacColl (as it did Lewis[38]) to the so-called paradoxes of strict implication.

THEOREM 5.1. [39]

1. $a : \epsilon = \epsilon$;

2. $\eta : a = \epsilon$;

3. $b^{\epsilon} : (a : b) = \epsilon$;

4. $a^{\eta} : (a : b) = \epsilon$.

Proof.

1. $a : \epsilon = (a' + \epsilon)^{\epsilon} = \epsilon^{\epsilon} = \epsilon$.
2. $\eta : a = (\eta a')^{\eta} = \eta^{\eta} = \epsilon$.
3. $b \leq a' + b$, so $b^{\epsilon} \leq (a' + b)^{\epsilon}$ by Theorem 3.1 (3)
   $= a : b$.
   So $b^{\epsilon} : (a : b) = \epsilon$ by Lemma 4.1 (1).
4. $ab' \leq a$, so $a' \leq (ab')'$ by Theorem 2.4 (3)
   whence $a^{\eta} = a'^{\epsilon} \leq (ab')'^{\epsilon}$ by Theorem 3.1 (3)
   $= (ab')^{\eta} = a : b$.
   So $a^{\eta} : (a : b) = \epsilon$ by Lemma 4.1 (1). □

In fact, it is a mistake simply to identify Theorem 5.1 (1) and (2) with the paradoxes of strict implication. For all they say is that there is a maximum (weakest) proposition (ϵ) implied by all others, and a minimum (strongest) proposition (η) which implies all others. Such so-called "Church constants" can in fact be added conservatively to relevance logic, that is, they can be added without necessarily disturbing the relevance features of the implication relation between other propositions.[40] It is (3) and (4) from Theorem 5.1 which exhibit a wider

[38] Lewis 1918, pp. 335 ff.; Lewis and Langford 1932, pp. 175 ff.

[39] MacColl 1901, p. 143 (3); Spencer 1973, p. 57 (12), (13).

[40] Anderson and Belnap 1975, § 27.1.2.

spread of irrelevance, that any necessary proposition (not just ϵ) is implied by any other, and that any impossible proposition (not just η) implies any other.

In a letter to Russell, MacColl says that

> [i]t is true that in ordinary speech the conjunction *if* usually suggests some necessary relation between the two sentences it connects; but the exigencies of logic force us to adhere to our definition, $A : B = (AB')^{\eta}$ and disregard this suggested relation. (MacColl 1998, 19 July 1901)

But this is an overstatement. If we choose to adhere to MacColl's definition, the exigencies of logic do indeed force us to disregard the suggested relation. But we might choose to explore an alternative definition. Elsewhere, MacColl dismisses this as psychologism.[41] But his own example in the letter to Russell shows that it is not a fair charge. He instances three (large) numbers, a, b and c, where $ab \neq c$ but not obviously so. Nonetheless, urges MacColl, Russell should concede, even before calculating the product of a and b, that

$$\text{if } ab = c \text{ then } 2ab = 2c. \tag{5}$$

But he goes on to observe that what makes (5) true is the impossibility of the equation $ab = c$ (since $ab \neq c$). But that undercuts his demand that Russell concede (5) before calculating. Obviously, if $ab = c$ then $2ab = 2c$; whereas it is not clear that, say, if $ab = c$ then $2ab = 7c$. (For if $ab = c$ and $2ab = 7c$, then $c = 0$ and so $a = b = 0$ too.) MacColl starts his example by recognising the relevance of implication, even though he ends by denying it.

To avoid even the irrelevance of the paradoxes of strict implication, one has to take a further step not contemplated by MacColl or Lewis. The source of their failure here lies in the fact that ':' is not *dyadically* intensional. It is the modalization of a truth-function. The truth-function 'and not' is dyadic; but the modal operator 'is impossible' is monadic. Sugihara (1955) produced a matrix to sieve out maximal and minimal formulae in implications; and Meyer (1974)[42] showed that no modalization of a truth-function could capture implication in any logic contained in that characterized by the Sugihara matrix, *viz RM*.[43] Not until implication is introduced by a truly dyadic intensional operator can the paradoxes of strict implication be excluded.

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[41] MacColl 1906, §§ 77–8, pp. 81–3.

[42] Cf. Anderson and Belnap 1975, § 29.12.

[43] See also Anderson and Belnap 1975, § 27.1.1.

Definition 5.1. *A* semi-group *consists of a set of elements closed under an associative operation, $\circ$ (fusion): $\langle A, \circ\rangle$ such that*

$$a \circ (b \circ c) = (a \circ b) \circ c \quad \textit{for all } a, b, c \in A. \qquad \textit{(Associativity for } \circ)$$

Definition 5.2. *A* monoid *is a semi-group with an identity, ϵ: $\langle A, \circ, \epsilon\rangle$ such that*

$$a \circ \epsilon = a = \epsilon \circ a. \qquad \textit{(Identity)}$$

Definition 5.3. *A semi-group is* commutative *if*

$$a \circ b = b \circ a. \qquad \textit{(Commutativity for } \circ)$$

Definition 5.4. *A* lattice-ordered semi-group *consists of a set of elements closed under meet, join and fusion: $\langle A, ., +, \circ\rangle$ such that $\langle A, ., +\rangle$ is a lattice, $\langle A, \circ\rangle$ is a semi-group and*

$$a \circ (b + c) = a \circ b + a \circ c \textit{ for all } a, b, c \in A \qquad \textit{(Distribution of } \circ \textit{ over } +)$$

Definition 5.5. *A lattice-ordered semi-group A is* residuated *if $\forall a, b \in A, \exists x, y \in A$ such that*

$$\forall c \in A, c \leq x \textit{ iff } c \circ a \leq b$$

and

$$\forall c \in A, c \leq y \textit{ iff } a \circ c \leq b.$$

We write $x = a \to b$ and $y = b \leftarrow a$.

Lemma 5.1. *If a lattice-ordered semi-group is commutative, $a \to b = b \leftarrow a$.*

Proof. $c \leq a \to b$ iff $c \circ b \leq a$ iff $b \circ c \leq a$ iff $c \leq b \leftarrow a$. So by Lemma 2.4, $a \to b = b \leftarrow a$. □

Definition 5.6. *A lattice-ordered semi-group A is* square-increasing *if $a \leq a \circ a$ for all $a \in A$.*

Definition 5.7.[44] *A* De Morgan monoid $\langle A, ., +, ', \circ, \epsilon\rangle$ *consists of a set of elements closed under meet, join, complement and fusion, such that $\langle A, ., +, \circ\rangle$ is a commutative square-increasing lattice-ordered monoid, the lattice $\langle A, ., +\rangle$ is distributive and for all $a, b \in A$:*

$$\textit{if } a \leq b \textit{ then } b' \leq a' \qquad \textit{(Contraposition)}$$

$$a'' = a \qquad \textit{(Double Negation)}$$

and

$$a \circ b \leq c \textit{ iff } b \circ c' \leq a' \textit{ iff } c' \circ a \leq b' \qquad \textit{(Antilogism)}$$

[44] Anderson and Belnap 1975, § 28.2.

LEMMA 5.2. *In a De Morgan monoid*

$$(a+b)' = a'.b'$$

Proof. Since $a \leq a+b$ and $b \leq a+b$,

$$(a+b)' \leq a' \text{ and } (a+b)' \leq b' \quad \text{by Contraposition}$$

so $(a+b)' \leq a'.b'$.

Conversely, since $a'.b' \leq a'$ and $a'.b' \leq b'$,

$$a = a'' \leq (a'.b')' \text{ and } b = b'' \leq (a'.b')' \quad \text{by Contraposition}$$

so $a+b \leq (a'.b')'$, whence $a'.b' = (a'.b')'' \leq (a+b)'$.

So $(a+b)' = a'.b'$. □

THEOREM 5.2. *Each De Morgan monoid is residuated.*

Proof. Take $a, b \in A$, the De Morgan monoid. Then $\forall c \in A$,

$c \circ a \leq b$ iff $a \circ b' \leq c'$ by Antilogism

iff $c \leq (a \circ b')'$ by Contraposition

Hence $a \to b = (a \circ b')'$

$= b \leftarrow a$ by Lemma 5.1, since A is commutative. □

De Morgan monoids give the algebraic structure of the logic of relevant implication, R, where the residual $a \to b$ expresses relevant implication. The logic $\mathrm{R}^{\Box}$ adds to R an S4-necessity. In $\mathrm{R}^{\Box}$, a modal relevant implication (entailment), $a \mathrel{\Box\!\!\rightarrow} b$, can be defined as $\Box(a \to b)$, equivalently, $(a \circ b')^{\eta}$, where $a^{\eta} =_{df} \Box(a')$, i.e., a'^{ϵ}. The algebra of $\mathrm{R}^{\Box}$ adds to De Morgan monoids a closure operation, $^{\pi}$ (possibility), as in the modal algebras above.[45]

DEFINITION 5.8. *A* modal l-monoid $\langle A, ., +, ', ^{\pi}, \circ, \epsilon\rangle$ *consists of a set of elements closed under meet, join, complement, possibility and fusion, such that* $\langle A, ., +, ', \circ, \epsilon\rangle$ *is a De Morgan monoid, and*

$$(a+b)^{\pi} = a^{\pi} + b^{\pi} \tag{K}$$

$$a \leq a^{\pi} \tag{T}$$

$$\eta^{\pi} = \eta \tag{N}$$

$$a^{\pi\pi} \leq a^{\pi} \tag{4}$$

and

$$a^{\pi} \circ b^{\epsilon} \leq (a \circ b)^{\pi} \tag{MP}$$

where $\eta = \epsilon'$ *and* $a^{\epsilon} = a'^{\pi}{}'$.

One could of course drop the postulate (4) if one wanted an algebra without reduction theses, more in MacColl's tradition.

[45] Anderson and Belnap 1975, § 28.2.5.

Theorem 5.3.

1. *Recall the definition of* $a \supset b$ *as* $a' + b$*. It follows that*

$$(a \supset b)^\epsilon \leq a^\epsilon \supset b^\epsilon;$$

2. *(MP) entails that*

$$(a \to b)^\epsilon \leq a^\epsilon \to b^\epsilon.$$

Proof.

1. The proof of Lemma 4.2 remains sound.
2. Recall that $a \to b = (a \circ b')'$. From (MP) we have (with b' for a and a for b)

$$b'^\pi \circ a^\epsilon \leq (b' \circ a)^\pi.$$

So $(a \to b)^\epsilon = (a \circ b')'^\epsilon = (a \circ b')^{\pi\prime}$
$\leq (b'^\pi \circ a^\epsilon)' = (a^\epsilon \circ b^{\epsilon\prime})' = a^\epsilon \to b^\epsilon$. □

We can show, by use of the following modal l-monoid, N, that the paradoxes of strict implication are invalidated. Let N be based on the same Boolean algebra as M, with the operation $^\pi$ as before, but now defining $\circ$ independently:

$a \circ b$	ϵ	x	x'	η	a^π
ϵ	ϵ	ϵ	ϵ	η	ϵ
x	ϵ	x	x'	η	ϵ
x'	ϵ	x'	ϵ	η	x'
η	η	η	η	η	η

Then $\to$ and $^\epsilon$ are given by the tables:

$a \to b$	ϵ	x	x'	η	a^ϵ
ϵ	ϵ	η	η	η	ϵ
x	ϵ	x	x'	η	x
x'	ϵ	η	x	η	η
η	ϵ	ϵ	ϵ	ϵ	η

Consequently, in N, let $a = x'$ and $b = x$. Then $b^\epsilon \to (a \to b)^\epsilon = x^\epsilon \to (x' \to x)^\epsilon = x \to \eta^\epsilon = x \to \eta = \eta$. Similarly, $a^\eta \to (a \to b)^\epsilon = \eta$ for the same assignment to a and b. Thus the paradoxes of strict implication are avoided in $\mathrm{R}^\Box$.

6. Conclusion

In this paper, I have attempted to exhibit some of the richness of MacColl's logic by recasting it in terms of the modal algebras (closure and extension algebras) of McKinsey, Tarski and Lemmon. Within this algebraic framework, I have derived many of MacColl's characteristic theses. Three of his theses, in particular, regarding necessity ($^{\epsilon}$) show that MacColl's logic was in fact the logic now known as T later introduced independently (of MacColl) by Feys (1937–1938) (his system t of § 28) and von Wright (1951) (his system M of Appendix II). It is interesting to speculate whether it was not the fact that MacColl was working within the Boole-Schröder algebraic paradigm which led him to normality ($\eta^{\pi} = \eta$, equivalently, $\epsilon^{\epsilon} = \epsilon$, von Wright's Rule of Tautology) and T, while Lewis' reformulation of modal logic within the proof-theoretic logistic of Frege, Peano and Russell took him away into the dead end of S1, S2 and S3.

I have explored only a small portion of MacColl's logic. There are many further original and fecund ideas remaining for investigation. I hope the framework I have developed here will prove a fruitful one for at least some of this exploration.

It should be noted, however, that I have left certain ideas deliberately unexplored because of an initial resistance to interpretation. Recall that I have used ϵ and η both as elements of the algebra and as operators (exponents). As I have used them, there is a systematic ambiguity. Nothing warranted use of the same symbol other than the two equations from Theorem 4.2:

$$a^{\epsilon} = a :: \epsilon$$

and

$$a^{\eta} = a :: \eta.$$

MacColl proceeds to use the connection between the element $a :: b$ and the equation $a = b$ expressed in Lemma 4.1 (2) to write these as[46]

$$a^{\epsilon} = (a = \epsilon) \tag{†}$$

and

$$a^{\eta} = (a = \eta) \tag{††}$$

and so to read a^{ϵ} not as an element but as expressing the validity of a, i.e., $a = \epsilon$, and similarly for η to express invalidity. (†) and (††) are

[46] MacColl 1901 pp. 143–4 (10) and (11).

ill-formed in my canon. For MacColl they support the identification of ϵ and η as element and exponent.

So far I can follow him, though only by the systematic ambiguity noted. However, MacColl proceeds to introduce θ as an element too, corresponding to the operator, $^{\theta}$. Thus he claims, for example,

$$\theta^{\epsilon} = \theta^{\eta} = \eta,$$ [47]

that is, it is impossible that a variable (contingent) element be either certain (ϵ) or impossible (η). It is an interesting question whether this idea can be so expressed in the language of modal algebras.

A different direction for research on MacColl's ideas would be to take further the suggestions I made in § 5, to develop an implication which is truly dyadic. The theory of modal l-monoids, combining the ideas of modal (i.e., closure and extension) algebras and De Morgan monoids, is largely unexplored.

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[47] MacColl 1901 p. 140.

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